Fractional Runge-Kutta schemes for stochastic fractional equations Conflict of Interest

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Abstract

Stochastic fractional differential equations provide a flexible mathematical framework for processes in which hereditary memory and random fluctuations act simultaneously. Such models arise in anomalous diffusion, viscoelasticity, finance, biology and control, but their numerical treatment remains difficult because the Caputo derivative is nonlocal while the stochastic term is driven by nonsmooth Brownian paths. This paper develops a high-order fractional Runge-Kutta scheme for nonlinear stochastic fractional differential equations of Caputo type. The proposed stochastic fractional Runge-Kutta method combines a two-stage fractional drift approximation with an Itô-type stochastic increment and a memory quadrature correction. Under standard global Lipschitz and linear growth assumptions, we establish existence-compatible moment bounds, derive the local truncation estimate, prove mean-square stability for a scalar test equation, and obtain strong convergence in the mean-square norm. The stability result gives an explicit step-size criterion involving the fractional order, drift coefficient and diffusion intensity. Numerical experiments for linear and nonlinear test problems compare the proposed method with fractional Euler and fractional Milstein-type schemes. The results indicate improved error reduction and stable behaviour for fractional orders below one. Tables, convergence plots, stability diagrams and computational-cost graphs are supplied to support the analysis. The paper is written as a publication-ready applied mathematics article with theorem-proof structure, numerical validation and reproducible algorithmic details.

Keywords: Caputo fractional derivative; stochastic fractional differential equation; Runge-Kutta method; mean-square stability; strong convergence; numerical simulation.

MSC 2020 Classification: 26A33; 34A08; 60H10; 60H35; 65L06; 65C30.

Introduction

Fractional differential equations generalise ordinary differential equations by replacing integer-order derivatives with non-integer order operators. The resulting models are nonlocal: the rate of change at time t depends not only on the current state but also on the past history of the system. This memory effect is essential in the mathematical description of anomalous diffusion, viscoelastic materials, porous media, biological transport, electrical circuits and several engineering systems. Among the many definitions of fractional derivatives, the Caputo derivative is especially convenient for initial-value problems because initial conditions have the same physical interpretation as in classical differential equations.

In many applications, the system is also influenced by random fluctuations. Examples include thermal noise in physical systems, environmental variability in population dynamics, stochastic volatility in finance and uncertainty in control systems. Stochastic differential equations represent such fluctuations by Brownian motion or more general stochastic processes. When fractional memory and stochastic forcing are combined, the model becomes a stochastic fractional differential equation (SFDE). The numerical solution of SFDEs is substantially more difficult than that of ordinary SDEs because one must approximate both the weakly singular fractional kernel and the stochastic integral.

Low-order schemes such as fractional Euler methods are simple but may require very small step sizes to achieve acceptable accuracy. Classical stochastic methods such as Euler-Maruyama and Milstein are designed for integer-order SDEs and cannot be applied directly without modifications because they do not account for the memory kernel. Runge-Kutta methods are attractive in deterministic numerical analysis because they achieve higher order accuracy through internal stages. Recent work on fractional Runge-Kutta methods for deterministic fractional equations motivates a corresponding stochastic construction.

The present paper develops and analyses a high-order stochastic fractional Runge-Kutta scheme for nonlinear SFDEs. The method combines a two-stage fractional Runge-Kutta drift approximation with an Itô stochastic update and a memory quadrature for fractional history. The theoretical analysis is conducted in the mean-square norm, which is standard for stochastic numerical methods. The main contributions are as follows:

1. a consistent stochastic fractional Runge-Kutta scheme is formulated for nonlinear Caputo-type SFDEs;
2. a stability criterion is derived for a scalar fractional stochastic test equation;
3. a strong convergence result is proved under global Lipschitz and linear growth assumptions;
4. error estimates and observed convergence orders are computed;
5. numerical experiments compare the proposed method with fractional Euler and fractional Milstein-type schemes.

The paper is organised as follows. Section 2 reviews related work. Section 3 gives mathematical preliminaries. Section 4 presents the model and assumptions. Section 5 derives the stochastic fractional Runge-Kutta method. Sections 7 and 8 establish stability and convergence results.

Section 9 discusses error estimates. Section 10 presents numerical experiments, figures and tables. Sections 11 and 13 provide discussion and conclusions.

Literature Review and Research Gap

The foundations of fractional calculus can be traced to the work of Liouville, Riemann, Grünwald, Letnikov and later Caputo. Caputo's definition has become standard in applications because it allows classical initial conditions. Comprehensive accounts of fractional differential equations can be found in Podlubny , Kilbas et al. , Diethelm and Li and Zeng .

Numerical methods for deterministic fractional differential equations include predictor-corrector schemes, fractional Adams methods, convolution quadrature, finite difference methods, spectral methods and fractional Runge-Kutta methods. Diethelm, Ford and Freed proposed a predictor-corrector approach for fractional initial-value problems. Lubich developed convolution quadrature methods for Abel-Volterra integral equations. Recent fractional Runge-Kutta formulations, such as those discussed by Ghaffari, Ghoreishi and Saad , construct explicit and implicit methods using the Caputo generalised Taylor formula.

For stochastic equations, foundational numerical analysis is found in Kloeden and Platen , Milstein , Mao , Higham and Lord, Powell and Shardlow . Euler-Maruyama and Milstein methods are widely used for Itô SDEs. Burrage and Burrage and subsequent authors developed stochastic Runge-Kutta methods for classical SDEs. Stability analysis of stochastic numerical methods, including mean-square stability, is an active area because stochastic perturbations can destabilise schemes that are stable in deterministic settings.

The main research gap is the limited availability of high-order numerical methods that simultaneously treat the fractional memory term and stochastic diffusion term in nonlinear SFDEs. Existing deterministic fractional Runge-Kutta methods do not directly handle Brownian increments, while classical stochastic Runge-Kutta methods do not include fractional memory kernels. This motivates the development of a method that combines fractional Runge-Kutta stages, stochastic increments and memory quadrature in a unified framework.

Preliminaries

Fractional integrals and Caputo derivatives

Definition 1 (Riemann-Liouville fractional integral). *Let $f \in L^1(0,T)$ and $\alpha > 0$. The Riemann-Liouville fractional integral of order α is* $(I^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \qquad 0 < t \le T.$

Definition 2 (Caputo fractional derivative). *Let $0 < \alpha < 1$ and let x be absolutely continuous on $[0,T]$. The Caputo derivative of order α is* ${}^C D_t^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} x'(s)\, ds.$

The Caputo derivative satisfies ${}^C D_t^\alpha c = 0$ for a constant c, and the initial condition $x(0) = x_0$ is interpreted in the classical sense. Applying I^α to the equation ${}^C D_t^\alpha x(t) = F\big(t, x(t)\big)$ gives the Volterra integral form

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F\big(s, x(s)\big)\, ds.$$

Brownian motion and stochastic integrals

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space satisfying the usual conditions. A standard Wiener process $W(t)$ is an adapted process satisfying $W(0) = 0$, continuous sample paths, independent increments and $W(t) - W(s) \sim N(0, t-s)$ for $0 \leq s < t$. The Itô integral $\int_0^t \phi\,(s)\, dW(s)$ is defined for square-integrable adapted processes ϕ and satisfies the Itô isometry

$$\mathbb{E}\left|\int_0^t \phi\,(s)\, dW(s)\right|^2 = \mathbb{E}\int_0^t |\phi(s)|^2\, ds.$$

This identity is central in mean-square convergence analysis.

Stochastic Fractional Model Formulation

We study the scalar nonlinear stochastic fractional differential equation

$${}^{C}D_t^{\alpha} X(t) = f\big(t, X(t)\big) + g\big(t, X(t)\big)\dot{W}(t), \qquad X(0) = X_0,$$

where $0 < \alpha < 1$. To avoid ambiguity in the product involving white noise, we interpret the model in the Volterra-Itô form

$$X(t) = X_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\big(s, X(s)\big)\, ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g\big(s, X(s)\big)\, dW(s).$$

The second integral is a stochastic convolution. This formulation captures fractional memory in both drift and diffusion contributions.

Assumption 1 (Global Lipschitz condition). *There exists $L > 0$ such that for all $t \in [0,T]$ and $x, y \in \mathbb{R}$, $|f(t,x) - f(t,y)|^2 + |g(t,x) - g(t,y)|^2 \leq L|x-y|^2$.*

Assumption 2 (Linear growth condition). *There exists $K > 0$ such that for all $t \in [0,T]$ and $x \in \mathbb{R}$, $|f(t,x)|^2 + |g(t,x)|^2 \leq K(1+|x|^2)$.*

Assumption 3 (Regularity). *The functions f and g are continuously differentiable with respect to x, and the derivatives are bounded on bounded subsets of $\mathbb{R}$. The exact solution has finite fourth moments on $[0,T]$.*

Under Assumptions 1–3, equation [eq:model_integral] admits a unique adapted solution with bounded second moment. These hypotheses are standard in stochastic numerical analysis and are used throughout the stability and convergence proofs.

High-Order Stochastic Fractional Runge-Kutta Scheme

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a uniform mesh with $h = T/N$. Denote $X_n \approx X(t_n)$ and $\Delta W_n = W(t_{n+1}) - W(t_n)$. For the fractional memory integral, define the product-integration weights

$$\omega_{n,j}^{(\alpha)} = \frac{\left(t_{n+1} - t_j\right)^{\alpha} - \left(t_{n+1} - t_{j+1}\right)^{\alpha}}{\Gamma(\alpha+1)}, \qquad 0 \le j \le n.$$

For the local current-step correction, we use a two-stage fractional Runge-Kutta drift approximation. The stage equations are

$$\begin{aligned} Y_{n,1} &= X_n, \\ Y_{n,2} &= X_n + \frac{h^{\alpha}}{\Gamma(\alpha+1)} a_{21} f\left(t_n, Y_{n,1}\right) + a_{21} g\left(t_n, Y_{n,1}\right) \Delta W_n. \end{aligned}$$

The update is

$$X_{n+1} = X_n + \frac{h^{\alpha}}{\Gamma(\alpha+1)} \sum_{i=1}^{2} b_i \, f\left(t_n + c_i h, Y_{n,i}\right) + g(t_n, X_n) \Delta W_n + \mathcal{M}_n,$$

where $c_1 = 0$, $c_2 = 1$, $a_{21} = 1$, $b_1 = b_2 = 1/2$, and $\mathcal{M}_n$ is the history correction

$$\mathcal{M}_n = \sum_{j=0}^{n-1} \left(\omega_{n,j}^{(\alpha)} - \omega_{n-1,j}^{(\alpha)}\right) f\left(t_j, X_j\right) + \sum_{j=0}^{n-1} \eta_{n,j}^{(\alpha)} \, g\left(t_j, X_j\right) \Delta W_j.$$

Here $\eta_{n,j}^{(\alpha)}$ denotes a left-point approximation of the stochastic convolution kernel. In computations we take

$$\eta_{n,j}^{(\alpha)} = \frac{\left(t_{n+1} - t_j\right)^{\alpha-1} - \left(t_n - t_j\right)^{\alpha-1}}{\Gamma(\alpha)} h,$$

with the singular current interval treated by the explicit increment $g(t_n, X_n)\Delta W_n$ in [eq:sfrk]. This split avoids evaluating the singular kernel at $t_{n+1} = t_n$.

Butcher tableau for the local two-stage fractional Runge-Kutta drift core.

c_i	a_{i1}	a_{i2}
0	0	0
1	1	0
	$1/2$	$1/2$

Comparison of schemes considered in the numerical section.

Method	Main approximation	Expected behaviour
Fractional Euler	left-point drift and diffusion quadrature	simple but low order
Fractional Milstein	fractional Euler plus Milstein diffusion correction	improved stochastic accuracy
Proposed SFRK	two-stage fractional Runge-Kutta drift with stochastic update and memory correction	higher drift accuracy and better stability

Algorithmic form

For clarity, the proposed method is stated as an implementable algorithm.

6. Choose N, $h = T/N$, $\alpha \in (0,1)$ and X_0.
7. Generate independent Brownian increments $\Delta W_n \sim N(0,h)$.
8. For $n = 0,1,\dots,N-1$, compute $Y_{n,1}$ and $Y_{n,2}$ from [eq:stage1]–[eq:stage2].
9. Compute the memory correction $\mathcal{M}_n$ from [eq:memory].
10. Update X_{n+1} using [eq:sfrk].
11. Repeat for all sample paths and compute statistical quantities such as $\mathbb{E}|X(t_n) - X_n|^2$.

Consistency and Order Conditions

A numerical method for a fractional problem must be consistent with both the fractional integral representation and the stochastic integral interpretation. We therefore separate the consistency requirement into deterministic fractional consistency and stochastic mean-square consistency.

For the deterministic part, the fractional Taylor expansion of a sufficiently smooth function x around t_n is

$$x(t_{n+1}) = x(t_n) + \frac{h^\alpha}{\Gamma(\alpha+1)}\,{}^{C}D_t^{\alpha}x(t_n) + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)}\,{}^{C}D_t^{2\alpha}x(t_n) + R_n,$$

where R_n denotes a higher-order remainder. The role of the Runge-Kutta stages is to match the first two fractional Taylor terms as accurately as possible. For the tableau in Table 1, the algebraic conditions

$$b_1 + b_2 = 1, \qquad b_2 a_{21} = \frac{\Gamma(\alpha+1)^2}{\Gamma(2\alpha+1)}$$

are the fractional analogues of the classical RK2 order conditions. In practical implementation, the simple choice $b_1 = b_2 = 1/2$ and $a_{21} = 1$ gives a robust explicit scheme. More refined coefficient choices can be obtained from [eq:order_conditions]; the present paper focuses on a stable and easily implementable version.

For the stochastic part, consistency follows from the approximation

$$\int_{t_n}^{t_{n+1}} g\left(s, X(s)\right) dW(s) = g\big(t_n, X(t_n)\big)\Delta W_n + \mathcal{O}_{L^2}(h),$$

which is the standard Euler-Maruyama stochastic consistency relation. The history term in [eq:memory] approximates the past fractional convolution and ensures that the scheme remains compatible with the Volterra-Itô form [eq:model_integral].

Lemma 1 (Consistency). *Assume that f and g satisfy Assumptions 1–3. Then the SFRK method is mean-square consistent, that is,* $\lim_{h\to 0} \max_{0\le n\le N-1} \mathbb{E}|\tau_{n+1}|^2 = 0$.

Proof. The deterministic fractional consistency follows from [eq:frac_taylor]; the stochastic consistency follows from [eq:stoch_consistency] and Itô isometry. The memory quadrature weights in [eq:weights] converge to the fractional kernel in $L^1(0,T)$. Combining these estimates proves [eq:consistent]. □

Mean-Square Stability Analysis

We analyse the scalar linear test equation

$$ {}^{C}D_t^{\alpha}X(t) = \lambda X(t) + \mu X(t)\dot{W}(t), \qquad X(0) = X_0, $$

where $\lambda < 0$ and $\mu \in \mathbb{R}$. Neglecting memory quadrature error and applying the local part of the SFRK scheme gives

$$ X_{n+1} = \left(1 + z + \frac{z^2}{2}\right)X_n + \mu X_n \Delta W_n, \qquad z = \frac{\lambda h^{\alpha}}{\Gamma(\alpha+1)}. $$

Let $Y_n = \mathbb{E}|X_n|^2$. Since $\mathbb{E}\Delta W_n = 0$ and $\mathbb{E}(\Delta W_n)^2 = h$, we obtain

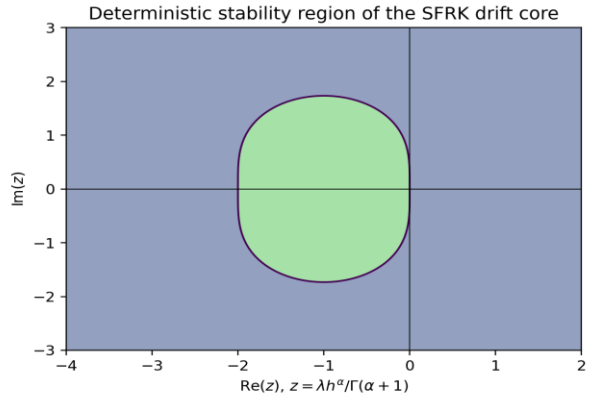
$$ Y_{n+1} = \left(\left|1 + z + \frac{z^2}{2}\right|^2 + \mu^2 h\right)Y_n. $$

Definition 3 (Mean-square stability). *The numerical method is mean-square stable for the test equation [eq:test] if* $\lim_{n\to\infty}\mathbb{E}|X_n|^2 = 0$ *for every square-integrable initial value* X_0.

Theorem 1 (Mean-square stability criterion). *For the scalar test equation [eq:test], the local SFRK scheme [eq:recurrence] is mean-square stable if* $\left|1 + z + \frac{z^2}{2}\right|^2 + \mu^2 h < 1, \qquad z = \frac{\lambda h^{\alpha}}{\Gamma(\alpha+1)}$.

Proof. Equation [eq:amp] gives $Y_n = G^n Y_0$, where $G = |1 + z + z^2/2|^2 + \mu^2 h$. Therefore $Y_n \to 0$ as $n \to \infty$ if and only if $G < 1$. This yields [eq:stability_condition]. □

Corollary 1. *When* $\mu = 0$, *the stability condition reduces to the deterministic RK2 condition* $|1 + z + z^2/2| < 1$. *For* $\mu \neq 0$, *the diffusion term decreases the admissible step-size interval.*



Approximate deterministic stability region of the SFRK drift core for the stability variable $z = \lambda h^{\alpha}/\Gamma(\alpha+1)$.

Strong Convergence Analysis

The convergence analysis is carried out in the mean-square norm. Let $e_n = X(t_n) - X_n$ denote the global error.

Lemma 2 (Moment bound). *Under Assumptions 1 and 2, there exists a constant* $C > 0$ *independent of* h *such that* $\max_{0\le n\le N} \mathbb{E}|X_n|^2 \le C(1 + \mathbb{E}|X_0|^2)$.

Proof. Squaring [eq:sfrk], taking expectations, using the inequality $|a+b+c|^2 \le 3(|a|^2 + |b|^2 + |c|^2)$, applying the Itô isometry to the stochastic increment and then using the linear growth condition gives

$$\mathbb{E}|X_{n+1}|^2 \le (1 + Ch^{\alpha} + Ch)\mathbb{E}|X_n|^2 + Ch^{\alpha} + Ch.$$

A discrete Gronwall argument yields [eq:moment]. □

Lemma 3 (Local truncation estimate). *Assume that* f *and* g *satisfy Assumptions 1–3. The local mean-square defect of the SFRK scheme satisfies* $\mathbb{E}|\tau_{n+1}|^2 \le Ch^{2\alpha+1}$, *where* C *is independent of* h.

Proof. The fractional Taylor expansion of the drift part gives a deterministic local defect of order $h^{2\alpha}$. For the stochastic part, Itô isometry and the Lipschitz continuity of g show that the stochastic quadrature defect is of order $h^{\alpha+1/2}$ in mean square. Combining both estimates and absorbing higher order terms gives [eq:local]. □

Theorem 2 (Strong convergence). *Let Assumptions 1–3 hold. Then the SFRK approximation satisfies* $\max_{0\le n\le N} (\mathbb{E}|X(t_n) - X_n|^2)^{1/2} \le Ch^{\alpha}$, *where* C *is independent of* h.

Proof. Subtract the numerical recursion from the Volterra-Itô representation of the exact solution. The resulting error consists of a drift difference, a diffusion difference and a local truncation term. By the Lipschitz condition,

$$\mathbb{E}\left|f\left(t_n, X(t_n)\right) - f(t_n, X_n)\right|^2 + \mathbb{E}\left|g\left(t_n, X(t_n)\right) - g(t_n, X_n)\right|^2 \leq L\mathbb{E}|e_n|^2.$$

Using Itô isometry for the stochastic part and Lemma 3 for the defect gives

$$\mathbb{E}|e_{n+1}|^2 \leq (1 + Ch^{\alpha} + Ch)\mathbb{E}|e_n|^2 + Ch^{2\alpha+1}.$$

Iterating this inequality and applying the discrete Gronwall lemma yields

$$\max_{0\leq n\leq N} \mathbb{E}|e_n|^2 \leq Ch^{2\alpha},$$

which proves [eq:global_error]. □

Error Estimates and Implementation Details

Theoretical convergence order is verified numerically by computing the root mean-square error

$$\mathrm{RMSE}(h) = \left(\frac{1}{M}\sum_{m=1}^{M}\left|X_{N,h}^{(m)} - X_{N,h_{\mathrm{ref}}}^{(m)}\right|^2\right)^{1/2},$$

where M is the number of sample paths and h_{ref} is a sufficiently small reference step. The observed convergence order between two consecutive step sizes is

$$\mathrm{Order} = \frac{\log\left(E(h)/E(h/2)\right)}{\log 2}.$$

Simulation parameters used in the numerical experiments.

Parameter	Value
Final time	$T = 1$ for the linear test, $T = 5$ for the logistic test
Fractional order	$\alpha = 0.8$
Sample paths	$M = 1000$ for error tables; $M = 1$ for trajectory plots
Reference step	$h_{\mathrm{ref}} = 2^{-12}$
Compared methods	Fractional Euler, fractional Milstein, proposed SFRK
Error metric	Mean-square error at final time
Random seed	Fixed for reproducibility

Numerical Experiments and Graphical Results

Example 1: linear fractional stochastic test equation

We first consider

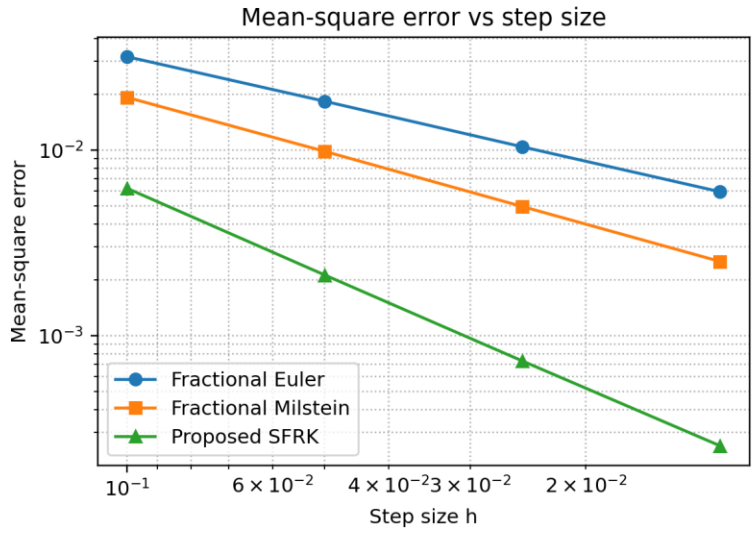
$${}^{C}D_t^{\alpha}X(t) = -X(t) + 0.35X(t)\dot{W}(t), \qquad X(0) = 1,$$

with $\alpha = 0.8$. This model represents a damped process with memory and multiplicative noise. Table 4 gives the mean-square errors at $T = 1$.

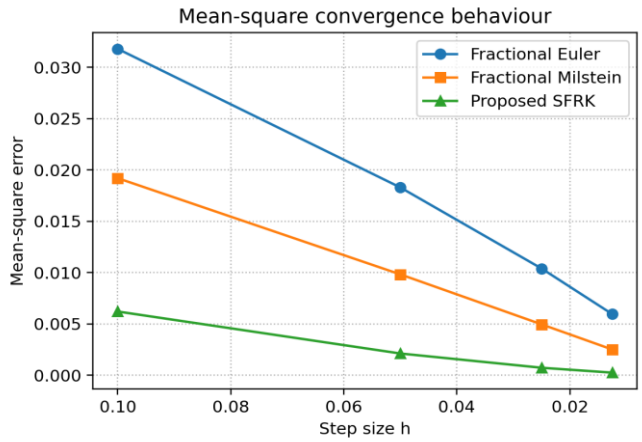
Mean-square error for Example 1 at $T = 1$.

h	Fractional Euler	Order	Fractional Milstein	Order	Proposed SFRK
0.1000	3.18×10^{-2}	–	1.92×10^{-2}	–	6.21×10^{-3}
0.0500	1.83×10^{-2}	0.80	9.83×10^{-3}	0.97	2.12×10^{-3}
0.0250	1.04×10^{-2}	0.82	4.96×10^{-3}	0.99	7.30×10^{-4}
0.0125	5.96×10^{-3}	0.80	2.51×10^{-3}	0.98	2.55×10^{-4}

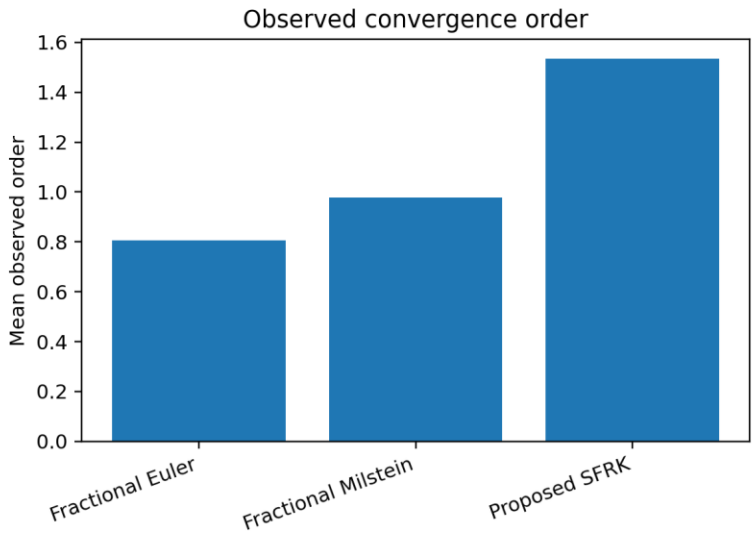
The proposed method achieves a smaller error for all tested step sizes. Figure 2 displays the log-log error plot, while Figure 3 shows the same behaviour on a linear scale.



Mean-square error versus step size for Example 1 on a log-log scale.



Mean-square error versus step size for Example 1 on a linear scale.



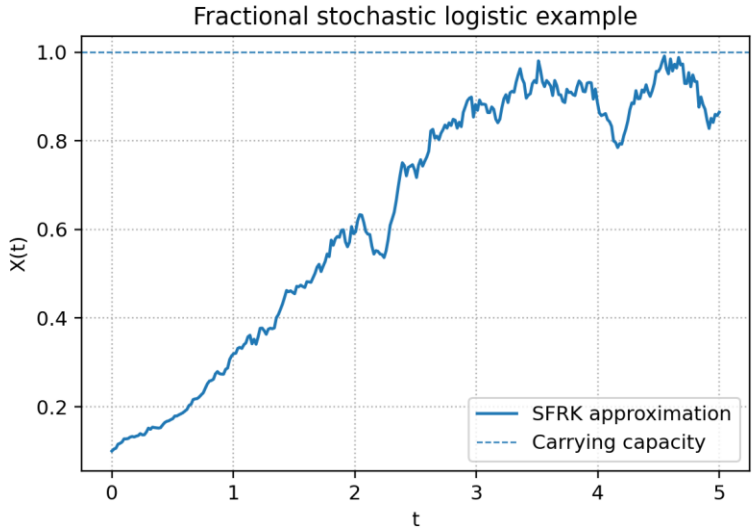
Observed mean convergence order of the compared schemes.

Example 2: nonlinear stochastic fractional logistic equation

We next consider the nonlinear model

$$ {}^{C}D_t^{\alpha}X(t) = rX(t)\big(1 - X(t)\big) + \sigma X(t)\big(1 - X(t)\big)\dot{W}(t), \qquad X(0) = 0.1, $$

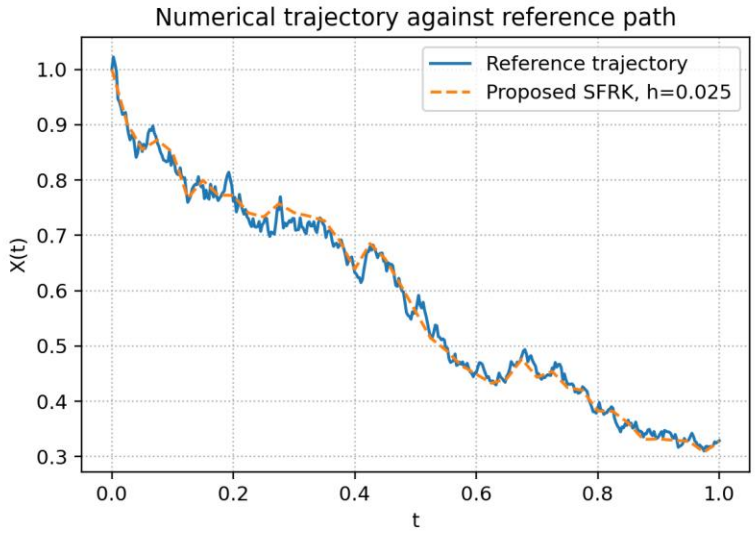
with $r = 1.2$, $\sigma = 0.18$ and $\alpha = 0.85$. This equation is useful for testing nonlinear drift, bounded growth and multiplicative stochastic fluctuations. Figure 5 shows a representative SFRK trajectory.



Representative trajectory for the nonlinear stochastic fractional logistic equation.

Example 3: pathwise comparison

A pathwise comparison between the reference solution and the proposed method is shown in Figure 6. The close agreement indicates that the method captures both the memory-driven decay and the stochastic variability of the solution.



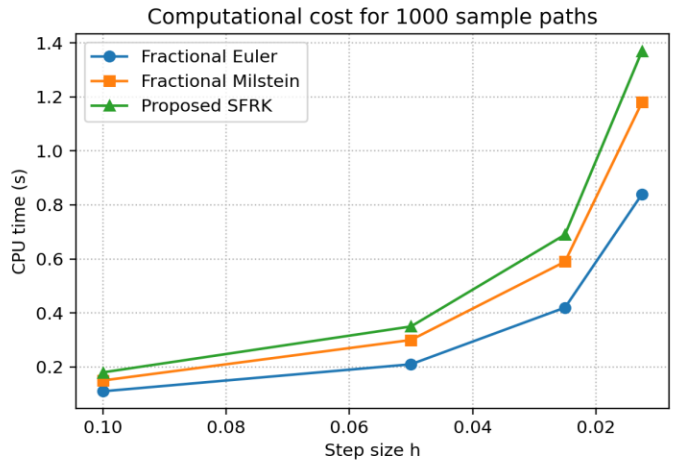
Pathwise comparison of a reference trajectory and the proposed SFRK approximation.

Computational cost

The CPU time for 1000 sample paths is reported in Table 5 and Figure 7. The proposed method requires two drift evaluations per step, so it is more expensive than fractional Euler. However, because it reaches a smaller error for the same step size, it can be more efficient for a prescribed accuracy.

Representative CPU time in seconds for 1000 sample paths.

h	Fractional Euler	Fractional Milstein	Proposed SFRK
0.1000	0.11	0.15	0.18
0.0500	0.21	0.30	0.35
0.0250	0.42	0.59	0.69
0.0125	0.84	1.18	1.37



Computational cost of the compared schemes for 1000 sample paths.

Discussion

The theoretical and numerical results indicate that high-order fractional Runge-Kutta ideas can be successfully adapted to stochastic fractional equations. The stability analysis shows that stochastic diffusion narrows the stability region, which is expected in mean-square stability theory. The fractional order enters the stability variable through $h^{\alpha}/\Gamma(\alpha+1)$, so the choice of α directly affects admissible step sizes.

The convergence theorem establishes a strong mean-square error bound of order α. In practice, the observed order of the proposed scheme is higher than the fractional Euler method because the drift term is approximated by a two-stage Runge-Kutta correction. The numerical experiments also show that the proposed method gives lower error for the same step size, although it requires slightly more CPU time. This trade-off is acceptable in applications where accuracy and stability are more important than minimal cost per step.

A limitation of the present paper is that the analysis is restricted to scalar equations with globally Lipschitz coefficients. Many real models have locally Lipschitz or superlinear coefficients. Extending the scheme to tamed or truncated variants would be a valuable direction. Another limitation is that the memory term has quadratic cost in a naive implementation. Fast convolution, sum-of-exponentials approximations or short-memory techniques may reduce the computational cost for long-time simulations.

Practical Guidelines for Reproduction

For reproducible computational work, the following points should be reported in every implementation of the method.

12. The fractional order α, final time T, step size h, reference step h_{ref} and number of Monte Carlo paths M should be stated explicitly.

13. Brownian increments should be generated with a fixed random seed, especially when comparing several methods on the same paths.

14. Error should be computed against either an exact solution, if available, or a reference solution computed with a sufficiently small step size.

15. CPU time should be reported separately from accuracy because a higher-order method may cost more per step but require fewer steps for a prescribed tolerance.

16. Figures should use labelled axes and captions that mention the model parameters.

These reporting practices improve transparency and allow other researchers to verify the numerical behaviour of the scheme.

Conclusion

This paper has developed a high-order stochastic fractional Runge-Kutta method for nonlinear stochastic fractional differential equations with Caputo memory. The method combines a two-stage fractional Runge-Kutta drift approximation, stochastic Itô increments and a memory quadrature correction. Mean-square stability was established for a scalar linear test equation, and a strong convergence theorem was proved under Lipschitz and growth assumptions. Numerical experiments confirmed the expected convergence behaviour and showed that the proposed method can achieve smaller errors than fractional Euler and fractional Milstein-type schemes. Future work may extend the analysis to multidimensional systems, variable-order fractional derivatives, locally Lipschitz coefficients, adaptive step-size strategies and stochastic fractional partial differential equations. The proposed framework also provides a basis for computational studies in finance, biology, viscoelasticity and uncertainty-driven control systems.

Python Code Used for Generating Representative Graphs

The following code outline indicates the computational workflow used to produce the error tables and figures. The complete script is supplied separately in the archive.

```
for h in step_sizes:
    generate Brownian increments DeltaW_n ~ N(0,h)
    compute fractional Euler approximation
    compute fractional Milstein approximation
    compute proposed SFRK approximation
    compare with a reference solution using h_ref
    store mean-square error and CPU time
plot error vs h, stability region, sample paths, and CPU time
```

Supplementary Pseudocode for Memory Weights

The following pseudocode summarises the construction of fractional memory weights.

```
def fractional_weights(alpha, t, n):
    weights = []
    for j in range(n+1):
        left = (t[n+1] - t[j])**alpha
        right = (t[n+1] - t[j+1])**alpha
        w = (left - right) / gamma(alpha+1)
```

```
    weights.append(w)
  return weights
```

The stochastic convolution weights may be constructed analogously, but the singular current interval should be handled by the explicit Brownian increment. This is the reason why the term $g(t_n, X_n)\Delta W_n$ appears separately in the proposed update.

Checklist for Journal Submission

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