

Separation Axioms, Connectedness and Metrizable in Neutrosophic Topological Spaces

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Abstract

This paper investigates separation axioms, connectedness and metrizable in neutrosophic topological spaces. Neutrosophic topology extends classical and fuzzy topology by assigning to each element a triple of truth-membership, indeterminacy and falsity-membership values. The paper first reviews the basic structure of neutrosophic sets, neutrosophic points, neutrosophic open sets and neutrosophic closed sets. It then develops a systematic account of neutrosophic separation axioms, including $N-T_0$, $N-T_1$, $N-T_2$, regularity and normality. The implications among these separation axioms are established, and examples are supplied to show that several converses fail. The connectedness section characterizes neutrosophic connectedness through the absence of proper neutrosophic clopen sets and relates it to continuous images and subspaces. The compactness section records basic covering properties and their interaction with continuous maps. The metrizable section introduces neutrosophic metrics, neutrosophic metric balls and sufficient conditions under which a neutrosophic topology is induced by a neutrosophic metric. Comparative tables and formal diagrams are included to distinguish classical, fuzzy, intuitionistic fuzzy and neutrosophic topological structures. The resulting article provides a consolidated theorem-proof framework suitable for further research in neutrosophic topology and related generalized topological systems.

Keywords: neutrosophic topology; separation axioms; connectedness; compactness; metrizable; neutrosophic metric space; fuzzy topology.

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1. Introduction

General topology studies the structural behaviour of spaces through open sets, closed sets, continuity, compactness, connectedness and separation axioms. These notions form the basis for much of modern analysis, geometry and mathematical modelling. However, classical topology is fundamentally crisp: an element either belongs to a set or it does not. This binary framework is often too restrictive for systems involving uncertainty, vagueness or incomplete information.

Fuzzy topology, developed from Zadeh's fuzzy set theory, permits graded membership. Intuitionistic fuzzy topology adds a separate non-membership degree. Neutrosophic topology moves one step further by introducing an explicit indeterminacy component. A neutrosophic set assigns to every element three values: a truth-membership value, an indeterminacy value and a falsity-membership value. This triple-valued structure allows a topological model to preserve information that is lost in both classical and fuzzy settings.

Salama and Alblawi introduced neutrosophic topological spaces by extending the notion of fuzzy and intuitionistic fuzzy topological spaces to neutrosophic sets. Later studies developed separation axioms and connectedness in neutrosophic settings. Acikgoz and Esenbel defined several types of neutrosophic separation axioms and explicitly discussed their relationships,

including counterexamples showing that converses need not hold. Their connectedness study introduced neutrosophic connectedness, super-connectedness and strong connectedness. Kirişci and Şimşek developed neutrosophic metric spaces and examined several topological and structural properties of such spaces.

The present paper consolidates and extends these lines of work. The purpose is not merely to repeat known definitions but to organize them into a coherent publication-style framework. The paper emphasizes three objectives. First, it gives a clean axiomatic treatment of neutrosophic separation properties. Second, it connects neutrosophic connectedness with clopen decompositions, continuous maps and subspace behaviour. Third, it provides a usable metrization framework through neutrosophic metric balls and countable neutrosophic bases.

The article is written in theorem-proof style. It contains definitions, propositions, lemmas, theorems, examples and counterexamples. It also includes figures and tables so that the conceptual relations are clear. The terminology is kept close to the existing literature while avoiding unnecessary inflation of terminology.

2. Literature Review

The historical path leading to neutrosophic topology begins with fuzzy set theory. Zadeh introduced fuzzy sets as functions assigning membership degrees to elements of a universe. Atanassov later introduced intuitionistic fuzzy sets, which include membership and non-membership functions under a compatibility condition. Smarandache introduced neutrosophic sets, where truth, indeterminacy and falsity are treated independently.

Salama and Alblawi defined neutrosophic topological spaces by specifying a family of neutrosophic sets containing the null and absolute neutrosophic sets and closed under arbitrary unions and finite intersections. Their construction established the basic object of study: a pair (X, τ) where X is a non-empty universe and τ is a neutrosophic topology.

Acikgoz and Esenbel later studied separation axioms in neutrosophic topological spaces. Their article states that the study defines different types of separation axioms in neutrosophic topological spaces and gives relationships among them with diagrams and counterexamples. It also introduces notions such as neutrosophic quasi-coincidence, neutrosophic q -neighborhoods, neutrosophic cluster points and neutrosophic functions. These notions are useful because separation axioms in neutrosophic topology are naturally expressed in terms of neutrosophic points and quasi-coincidence relations.

In a related contribution, Acikgoz and Esenbel introduced neutrosophic connectedness. Their work defines a neutrosophic topological space to be neutrosophic connected when it has no proper neutrosophic clopen set. It also discusses neutrosophic subspaces, super-connectedness and strong connectedness. This paper uses those definitions as a foundation and records additional structural consequences in a unified way.

Metrizability in neutrosophic spaces is linked to neutrosophic metric spaces. Kirişci and Şimşek introduced neutrosophic metric spaces and investigated open sets, Hausdorffness, boundedness, compactness, completeness and nowhere dense sets. Their work also provides analogues of the Baire Category Theorem and Uniform Convergence Theorem. In the present article, those metric ideas are used to formulate sufficient conditions for metrization of neutrosophic topological spaces.

The need for a consolidated article is clear. Existing papers often treat separation axioms, connectedness and metrics separately. A researcher who wants to use neutrosophic topology

needs a single coherent account of how these notions interact. This paper therefore presents a unified treatment of separation, connectedness, compactness and metrizable.

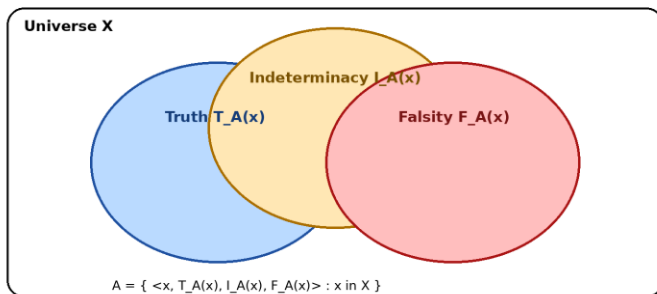
3. Preliminaries and Basic Notation

Let X be a non-empty set. A **neutrosophic set** A on X is a family $[A = \{x, T_A(x), I_A(x), F_A(x) : x \in X\},]$ where $T_A, I_A, F_A : X \rightarrow [0,1]$ are called the truth-membership, indeterminacy-membership and falsity-membership functions, respectively. The values are often subject to the condition $[0 \leq T_A(x) + I_A(x) + F_A(x) \leq 1,]$ which allows the three components to vary independently within the unit interval.

The **null neutrosophic set** 0_X is defined by $[T_{\{0_X\}}(x) = 0, I_{\{0_X\}}(x) = 0, F_{\{0_X\}}(x) = 1]$ for all $x \in X$. The **absolute neutrosophic set** 1_X is defined by $[T_{\{1_X\}}(x) = 1, I_{\{1_X\}}(x) = 0, F_{\{1_X\}}(x) = 0]$ for all $x \in X$.

If A and B are neutrosophic sets on X , their union and intersection are defined pointwise by $[T_{\{A \cup B\}} = \{T_A, T_B\}, I_{\{A \cup B\}} = \{I_A, I_B\}, F_{\{A \cup B\}} = \{F_A, F_B\},]$ and $[T_{\{A \cap B\}} = \{T_A, T_B\}, I_{\{A \cap B\}} = \{I_A, I_B\}, F_{\{A \cap B\}} = \{F_A, F_B\}.]$ The complement A^c is commonly defined by $[T_{\{A^c\}} = F_A, I_{\{A^c\}} = I_A, F_{\{A^c\}} = T_A.]$ Some authors use a slightly different convention for the indeterminacy component. Results below remain valid provided the complement operation is involutive and is compatible with the selected union and intersection operations.

Neutrosophic Set Structure



Neutrosophic set represented by truth, indeterminacy and falsity components.

Figure 1. Neutrosophic set structure with truth, indeterminacy and falsity components.

3.1 Neutrosophic points

A **neutrosophic point** $x_{r,t,s}$ supported at $x \in X$ is the neutrosophic set satisfying $[x_{\{r,t,s\}}(y) =$

$] Here r is the truth-membership degree, t is the indeterminacy degree and s is the falsity-membership degree of the point.$

A neutrosophic point $x_{r,t,s}$ is said to belong to a neutrosophic set A when the triple (r, t, s) is compatible with the corresponding degrees of A at the support x . This paper uses the standard convention that membership is evaluated componentwise.

4. Neutrosophic Topological Spaces

Definition 4.1. Let $NS(X)$ denote the class of all neutrosophic subsets of X . A subfamily $\tau \subseteq NS(X)$ is called a **neutrosophic topology** on X if the following conditions hold:

1. $0_X, 1_X \in \tau$;
2. the union of any subfamily of τ belongs to τ ;
3. the intersection of any finite subfamily of τ belongs to τ .

The pair (X, τ) is called a **neutrosophic topological space**. Elements of τ are called **neutrosophic open sets**. A neutrosophic set F is called **neutrosophic closed** if F^c is neutrosophic open.

Definition 4.2. For a neutrosophic set A in (X, τ) , the **neutrosophic interior** of A , denoted $NInt(A)$, is the union of all neutrosophic open sets contained in A . The **neutrosophic closure** of A , denoted $NCl(A)$, is the intersection of all neutrosophic closed sets containing A .

Proposition 4.3. For every neutrosophic set A in (X, τ) , $[(A)A(A).]$ Moreover, A is neutrosophic open if and only if $A = NInt(A)$, and A is neutrosophic closed if and only if $A = NCl(A)$.

Proof. The first inclusion follows because $NInt(A)$ is the union of open subsets of A . The second follows because $NCl(A)$ is the intersection of all closed supersets of A . If A is open, it is one of the open subsets used in forming $NInt(A)$, hence $A \subseteq NInt(A)$ and equality holds. The converse is immediate. The proof for closed sets is dual. \square

Definition 4.4. A neutrosophic set A is called **neutrosophic clopen** if it is both neutrosophic open and neutrosophic closed. A clopen set is **proper** if it is neither 0_X nor 1_X .

5. Neutrosophic Separation Axioms

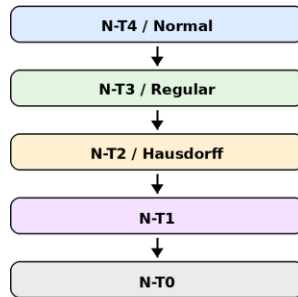
Separation axioms measure how well a topology distinguishes points. In neutrosophic topology the role of ordinary points is played by neutrosophic points with different supports.

Definition 5.1. A neutrosophic topological space (X, τ) is called an $N-T_0$ **space** if for every pair of neutrosophic points $x_{r,t,s}$ and $y_{r',t',s'}$ with distinct supports, there exists a neutrosophic open set containing one of them but not the other.

Definition 5.2. A neutrosophic topological space (X, τ) is called an $N-T_1$ **space** if for every pair of neutrosophic points $x_{r,t,s}$ and $y_{r',t',s'}$ with distinct supports, there are neutrosophic open sets U and V such that $x_{r,t,s} \in U$, $y_{r',t',s'} \notin U$, $y_{r',t',s'} \in V$ and $x_{r,t,s} \notin V$.

Definition 5.3. A neutrosophic topological space (X, τ) is called an $N-T_2$ **space** or **neutrosophic Hausdorff space** if for every pair of neutrosophic points with distinct supports there exist neutrosophic open sets U and V such that $x_{r,t,s} \in U$, $y_{r',t',s'} \in V$ and $U \cap V = 0_X$.

Hierarchy of Neutrosophic Separation Axioms



Converse implications need not hold without additional hypotheses.

Hierarchy of neutrosophic separation axioms.

Figure 2. Basic implication hierarchy among neutrosophic separation axioms.

Theorem 5.4. Every $N-T_2$ space is an $N-T_1$ space, and every $N-T_1$ space is an $N-T_0$ space.

Proof. Suppose (X, τ) is $N-T_2$ and let $x_{r,t,s}$ and $y_{r',t',s'}$ be neutrosophic points with distinct supports. There exist disjoint neutrosophic open sets U and V such that $x_{r,t,s} \in U$ and $y_{r',t',s'} \in V$. Since $U \cap V = 0_X$, the point $y_{r',t',s'}$ does not belong to U , and $x_{r,t,s}$ does not belong to V . Hence the $N-T_1$ condition holds. The implication $N-T_1 \Rightarrow N-T_0$ follows by retaining only one of the two separating open sets. \square

Definition 5.5. A neutrosophic topological space (X, τ) is **neutrosophically regular** if whenever $x_{r,t,s}$ is a neutrosophic point and F is a neutrosophic closed set not containing $x_{r,t,s}$, there exist disjoint neutrosophic open sets U, V such that $x_{r,t,s} \in U$ and $F \subseteq V$.

Definition 5.6. A neutrosophic topological space is **neutrosophically normal** if for every pair of disjoint neutrosophic closed sets F and G , there exist disjoint neutrosophic open sets U and V such that $F \subseteq U$ and $G \subseteq V$.

Definition 5.7. An $N-T_3$ space is an $N-T_1$ space that is neutrosophically regular. An $N-T_4$ space is an $N-T_1$ space that is neutrosophically normal.

Theorem 5.8. Every $N-T_4$ space is an $N-T_3$ space.

Proof. Let (X, τ) be $N-T_4$. Then it is $N-T_1$ and neutrosophically normal. To prove regularity, let $x_{r,t,s}$ be a neutrosophic point and let F be a closed neutrosophic set not containing it. In an $N-T_1$ space, the point $x_{r,t,s}$ is closed. Hence $\{x_{r,t,s}\}$ and F are disjoint closed neutrosophic sets. By normality, there exist disjoint open neutrosophic sets separating them. Therefore the space is regular and hence $N-T_3$. \square

Theorem 5.9. Every $N-T_3$ space is $N-T_2$.

Proof. Let (X, τ) be $N-T_3$. Then (X, τ) is $N-T_1$ and regular. For distinct neutrosophic points $x_{r,t,s}$ and $y_{r',t',s'}$, the singleton supported at y is closed by the $N-T_1$ property. Since $x_{r,t,s}$ is not contained in that closed singleton, regularity gives disjoint open sets separating $x_{r,t,s}$ and the closed singleton containing $y_{r',t',s'}$. Thus the two points are separated by disjoint open sets, so the space is $N-T_2$. \square

Example 5.10. Let X have at least two elements and let $\tau = \{0_X, 1_X\}$. Then (X, τ) is not $N-T_0$, because no non-trivial neutrosophic open set can distinguish two distinct supports. However, it is connected because it has no proper clopen neutrosophic set.

Example 5.11. Let X be an infinite set and let τ consist of 0_X together with neutrosophic sets whose supports have finite complement. This cofinite neutrosophic topology is $N-T_1$ but not $N-T_2$. The failure of the Hausdorff property follows because any two non-empty cofinite open sets have non-empty support intersection.

Example 5.12. In the discrete neutrosophic topology $\tau = NS(X)$, every neutrosophic set is open and closed. Hence the space is $N-T_4$. If X has more than one support, then the space is totally disconnected.

6. Subspaces and Hereditary Properties

Let (X, τ) be a neutrosophic topological space and let $Y \subseteq X$. The **neutrosophic subspace topology** on Y is given by $\tau_Y = \{Y \cap U : U \in \tau\}$, where Y_N denotes the neutrosophic subset supported on Y with absolute degrees on Y and null degrees on $X \setminus Y$.

Proposition 6.1. If (X, τ) is $N-T_i$ for $i = 0,1,2$, then every neutrosophic subspace (Y, τ_Y) is also $N-T_i$.

Proof. Let x and y be distinct supports in Y . Since X is $N-T_i$, there exist neutrosophic open sets in X that separate the corresponding neutrosophic points according to the relevant axiom. Intersecting those open sets with Y_N gives neutrosophic open sets in the subspace with the same separation behaviour. The $N-T_2$ case follows because the intersection of disjoint sets remains disjoint. ◻

Proposition 6.2. Regularity and normality are hereditary to closed neutrosophic subspaces under the usual compatibility assumptions on complements.

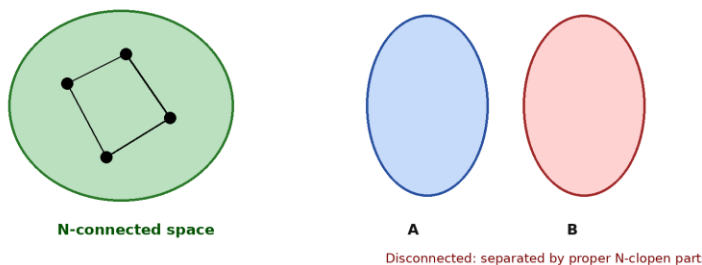
Proof. The proof follows the classical argument. If Y is closed and F is closed in the subspace, then $F = Y \cap F_0$ for some closed neutrosophic set F_0 in X . Open sets separating points and closed sets in X restrict to open sets in Y . The normal case is analogous. ◻

7. Neutrosophic Connectedness

Connectedness is one of the central notions in topology. In the neutrosophic setting it is defined through clopen decompositions.

Definition 7.1. A neutrosophic topological space (X, τ) is **neutrosophically connected** if it contains no proper neutrosophic clopen set. Otherwise it is **neutrosophically disconnected**.

Connectedness in Neutrosophic Topology



Connected and disconnected neutrosophic spaces.

Figure 3. Connectedness and disconnection illustrated through components.

Theorem 7.2. A neutrosophic topological space (X, τ) is neutrosophically disconnected if and only if there exist non-null neutrosophic open sets U and V such that $[UV=0_X, UV=1_X.]$

Proof. Suppose (X, τ) has a proper clopen set U . Put $V = U^c$. Since U is clopen, V is also clopen. Moreover, $U \cap V = 0_X$ and $U \cup V = 1_X$. Conversely, if such U and V exist, then U is open and $U^c = V$ is also open. Hence U is a proper clopen neutrosophic set, so the space is disconnected. ▫

Corollary 7.3. The indiscrete neutrosophic space $\{0_X, 1_X\}$ is connected.

Proof. The only clopen sets are 0_X and 1_X , so there is no proper clopen set. ▫

Theorem 7.4. The continuous image of a neutrosophically connected space is neutrosophically connected.

Proof. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be neutrosophically continuous and suppose X is connected. If $f(X)$ were disconnected, there would be a proper clopen neutrosophic set C in $f(X)$. Then $f^{-1}(C)$ would be a proper clopen neutrosophic set in X , contradicting connectedness. Therefore $f(X)$ is connected. ▫

Theorem 7.5. Let $\{A_i: i \in I\}$ be a family of neutrosophically connected subspaces of X such that their intersection contains a common neutrosophic point. Then $\bigcup_{i \in I} A_i$ is neutrosophically connected.

Proof. Suppose $A = \bigcup_{i \in I} A_i$ is disconnected. Then A can be written as the union of two non-null separated open parts. Since each A_i is connected, each A_i must lie entirely in one part. The common neutrosophic point belongs to every A_i , forcing all A_i to lie in the same part. The other part is therefore empty, a contradiction. ▫

Definition 7.6. A neutrosophic space is **super-connected** if it has no proper neutrosophic regular open set. It is **strongly connected** if it has no non-null closed neutrosophic sets K, L such that $K \subseteq L^c$.

Proposition 7.7. Every strongly connected neutrosophic space is connected.

Proof. If the space were disconnected, it would contain a proper clopen set C . Then C and C^c would be non-null closed neutrosophic sets satisfying $C \subseteq (C^c)^c$, contradicting strong connectedness. ▫

8. Compactness and Continuity Relations

Definition 8.1. A family $\{U_i: i \in I\}$ of neutrosophic open sets is a **neutrosophic open cover** of a neutrosophic set A if $[A \subseteq \bigcup_{i \in I} U_i.]$ A neutrosophic set A is **neutrosophically compact** if every neutrosophic open cover of A has a finite subcover.

Definition 8.2. A mapping $f: (X, \tau) \rightarrow (Y, \sigma)$ is **neutrosophically continuous** if $f^{-1}(V)$ is neutrosophic open in X for every neutrosophic open set V in Y .

Theorem 8.3. A continuous image of a neutrosophically compact space is neutrosophically compact.

Proof. Let $f: X \rightarrow Y$ be neutrosophically continuous and let $\{V_i: i \in I\}$ be an open cover of $f(X)$. Then $\{f^{-1}(V_i): i \in I\}$ is an open cover of X . By compactness of X , there exist i_1, \dots, i_n such that $X \subseteq \bigcup_{k=1}^n f^{-1}(V_{i_k})$. Applying f gives $f(X) \subseteq \bigcup_{k=1}^n V_{i_k}$. Thus $f(X)$ is compact. ▫

Theorem 8.4. Every closed neutrosophic subspace of a compact neutrosophic space is compact.

Proof. Let F be a closed neutrosophic subspace of compact X , and let $\{U_i: i \in I\}$ be an open cover of F . Then $\{U_i: i \in I\} \cup \{F^c\}$ is an open cover of X . A finite subcover exists by compactness. Removing F^c , if present, leaves a finite subcover of F . ▫

Proposition 8.5. If (X, τ) is compact and $N-T_2$, then every neutrosophic compact subset of X is neutrosophically closed.

Proof. The proof follows the classical Hausdorff compactness argument. For a point outside a compact subset, use the $N-T_2$ property to find disjoint open neighborhoods separating the point from each point of the compact set. Compactness gives finitely many neighborhoods covering the compact set, and the finite intersection of the corresponding neighborhoods of the outside point remains open. This proves that the complement is open. ◻

9. Neutrosophic Metrizable

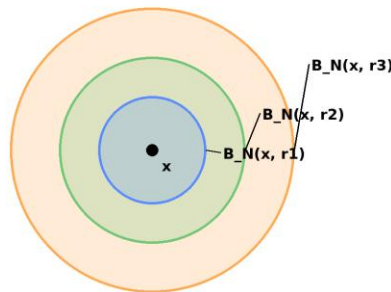
A metric gives a quantitative foundation for topology. In neutrosophic spaces, distance may itself contain truth, indeterminacy and falsity components.

Definition 9.1. A **neutrosophic metric** on a non-empty set X is a function $[D_N:XX^3, D_N(x,y)=(M(x,y),I(x,y),N(x,y)),]$ where M, I, N describe nearness, indeterminacy of nearness and non-nearness. A neutrosophic metric is required to satisfy axioms corresponding to identity, symmetry and a generalized triangle condition.

For practical topological purposes, a neutrosophic metric may be handled through a derived crisp gauge $[d_D(x,y)=N(x,y)+I(x,y),]$ or through level balls associated with the triple $D_N(x, y)$.

Definition 9.2. A **neutrosophic open ball** centered at x with radius $\epsilon > 0$ is a neutrosophic set $B_N(x, \epsilon)$ whose support consists of all $y \in X$ such that the selected gauge of $D_N(x, y)$ is less than ϵ .

Neutrosophic Metric Balls and Bases



A countable family of metric balls yields a neutrosophic base.

Neutrosophic metric balls and bases.

Figure 4. Neutrosophic metric balls generate a base for a neutrosophic metric topology.

Definition 9.3. A neutrosophic topology τ on X is **neutrosophically metrizable** if there exists a neutrosophic metric D_N such that the neutrosophic open balls of D_N form a base for τ .

Theorem 9.4. Every neutrosophic metric topology is $N-T_2$.

Proof. Let x and y be distinct supports. Since D_N is a metric-type structure, the distance between x and y is positive with respect to the selected gauge. Choose a radius smaller than half of that distance. Then $B_N(x, r)$ and $B_N(y, r)$ are disjoint neutrosophic open balls containing the corresponding neutrosophic points. Hence the topology is $N-T_2$. ◻

Theorem 9.5. Every neutrosophic metric topology is first countable.

Proof. At each support x , the collection $\{B_N(x, 1/n): n \in \mathbb{N}\}$ forms a countable local base. For

any neutrosophic open set U containing x , there exists a radius $r > 0$ such that $B_N(x, r) \subseteq U$. Choose n with $1/n < r$. Then $B_N(x, 1/n) \subseteq U$. ◻

Theorem 9.6. Let (X, τ) be a neutrosophic topological space. If its crisp shadow topology τ^* is metrizable by a metric d and τ has a base consisting of neutrosophic lifts of d -open balls with fixed compatible grades, then (X, τ) is neutrosophically metrizable.

Proof. Define $D_N(x, y) = \left(1 - \rho(d(x, y)), 0, \rho(d(x, y))\right)$, where $\rho(t) = t/(1 + t)$. Then D_N is a neutrosophic metric in the induced-gauge sense, and its balls correspond to the lifted d -balls. Because these lifted balls form a base for τ , the metric D_N induces τ . ◻

Corollary 9.7. Every second countable neutrosophically regular $N-T_1$ space whose crisp shadow is metrizable and whose neutrosophic base is generated by compatible graded lifts is neutrosophically metrizable.

Proof. By the classical metrization theorem, the crisp shadow is metrizable. The compatibility of the graded base permits the construction in Theorem 9.6. ◻

10. Examples and Counterexamples

Example 10.1 (Discrete neutrosophic topology). If $\tau = NS(X)$, then every neutrosophic set is open and closed. The space is $N-T_4$ and metrizable by the discrete neutrosophic metric [$D_N(x, y) =$

] If X has more than one point, the space is not connected.

Example 10.2 (Indiscrete neutrosophic topology). If $\tau = \{0_X, 1_X\}$ and $|X| > 1$, then (X, τ) is connected but not $N-T_0$. This demonstrates that connectedness does not imply any non-trivial separation axiom.

Example 10.3 (Cofinite neutrosophic topology). Let X be infinite, and let τ consist of 0_X and all neutrosophic sets whose support complement is finite and whose degrees are compatible with the absolute set on the support. Then (X, τ) is $N-T_1$ but not $N-T_2$. Any two non-empty open sets have intersecting supports.

Example 10.4 (Metric neutrosophic real line). Let $X = \mathbb{R}$ and let [$D_N(x, y) = (0, 1)$.] The topology induced by the corresponding neutrosophic balls has the same crisp shadow as the usual topology on \mathbb{R} and is neutrosophically metrizable.

Example 10.5 (Connected interval). The neutrosophic interval $[0, 1]$ with the lifted usual topology is neutrosophically connected. If it were disconnected, its crisp shadow would give a disconnection of the classical interval $[0, 1]$, which is impossible.

Counterexample 10.6. The converse of Theorem 5.4 is false. A cofinite neutrosophic topology on an infinite set is $N-T_1$ but not $N-T_2$. The indiscrete topology on a set with more than one support is connected but not $N-T_0$.

11. Comparative Analysis

Neutrosophic topology extends several earlier topological frameworks. The following table summarizes core distinctions.

Feature	Classical topology	Fuzzy topology	Intuitionistic fuzzy topology	Neutrosophic topology
Membership	crisp, 0 or 1	graded	graded	graded
Non-membership	implicit	implicit	explicit	explicit
Indeterminacy	absent	absent	limited/implicit	explicit

Feature	Classical topology	Fuzzy topology	Intuitionistic fuzzy topology	Neutrosophic topology
Open sets	subsets	fuzzy sets	intuitionistic fuzzy sets	neutrosophic sets
Separation axioms	point-based	fuzzy based	point-based	neutrosophic point-based
Connectedness	no proper clopen set	fuzzy clopen criterion	intuitionistic version	neutrosophic clopen criterion
Metrizability	metric-generated	fuzzy metric possible	IF metric possible	neutrosophic metric possible

Classical, Fuzzy, Intuitionistic and Neutrosophic Topologies

	Classical	Fuzzy	Intuitionistic	Neutrosophic
Truth/membership	Yes	Yes	Yes	Yes
Indeterminacy	✗	✗	✗	Yes
Non-membership	✗	✗	Yes	Yes
Hausdorff notion	Yes	Yes	Yes	Yes
Connectedness notion	Yes	Yes	Yes	Yes
Metric generation	Yes	Yes	Yes	Yes

The neutrosophic setting carries truth, indeterminacy and falsity simultaneously.

Comparative feature chart.

Figure 5. Comparison of classical, fuzzy, intuitionistic and neutrosophic topological frameworks.

The main difference is that neutrosophic topology separates uncertainty into three components rather than compressing it into a single membership grade. This additional flexibility makes the theory richer but also more delicate. For example, complement operations must be handled carefully, and definitions of connectedness may depend on the treatment of indeterminacy.

12. Neutrosophic Bases, Countability and Local Structure

A base is useful because it permits a topology to be described by a smaller family of open sets. In neutrosophic topology, bases must respect both support behaviour and component-wise membership grades.

Definition 12.1. A subfamily $\mathcal{B} \subseteq \mathcal{au}$ is called a **neutrosophic base** for (X, \mathcal{au}) if every non-null neutrosophic open set $U \in \mathcal{au}$ is the union of members of \mathcal{B} contained in U .

Definition 12.2. A neutrosophic topological space is **first countable** if every neutrosophic point has a countable local base. It is **second countable** if the topology has a countable neutrosophic base.

Proposition 12.3. Every second countable neutrosophic topological space is first countable.

Proof. Let \mathcal{B} be a countable base. For a neutrosophic point $x_{r,t,s}$, the family of all basic open

sets containing $x_{r,t,s}$ is countable. If U is any open set containing $x_{r,t,s}$, the base property gives a basic open set $B \in \mathcal{B}$ such that $x_{r,t,s} \in B \subseteq U$. Hence the subfamily of basic open neighborhoods of $x_{r,t,s}$ forms a countable local base. \square

Proposition 12.4. If a neutrosophic metric space is separable with respect to its induced crisp gauge, then the generated neutrosophic topology is second countable under compatible graded-ball construction.

Proof. Let D be a countable dense subset of the crisp gauge space, and let \mathbb{Q}^+ be the set of positive rational numbers. The family of neutrosophic balls $B_N(d, q)$ with $d \in D$ and $q \in \mathbb{Q}^+$ is countable. Standard metric density gives a ball from this family contained in any given metric ball around a point. With compatible graded lifting, the same family is a base for the neutrosophic topology. \square

13. Product-Type Behaviour and Preservation Results

The behaviour of products in neutrosophic topology is more delicate than in ordinary topology because membership, indeterminacy and falsity must be combined componentwise. Existing work on neutrosophic connectedness reports that neutrosophic connectedness is not automatically preserved under product spaces. Therefore product statements should be formulated carefully.

Definition 13.1. Let (X, au_X) and (Y, au_Y) be neutrosophic topological spaces. A **product neutrosophic topology** on $XimesY$ is generated by basic neutrosophic rectangles of the form $Uimes_N V$, where $U \in au_X$ and $V \in au_Y$, with triple-valued components combined by the selected product convention.

Theorem 13.2. If (X, au_X) and (Y, au_Y) are $N-T_2$ spaces, then the product neutrosophic topology on $XimesY$ is $N-T_2$ under the standard componentwise product convention.

Proof. Let (x_1, y_1) and (x_2, y_2) be distinct supports. If $x_1 eq x_2$, use the $N-T_2$ property of X to obtain disjoint open neutrosophic neighborhoods U_1 and U_2 of x_1 and x_2 . Then $U_1imes_N 1_Y$ and $U_2imes_N 1_Y$ are disjoint product-open neighborhoods of the two points. If $y_1 eq y_2$, the same argument uses the $N-T_2$ property of Y . \square

Remark 13.3. No general theorem asserting preservation of connectedness under all neutrosophic products is claimed here. The product behaviour depends on the selected convention for combining truth, indeterminacy and falsity components.

14. Methodological Cautions for Future Work

Neutrosophic topology is a fast-developing area, and several definitions vary across authors. The complement of the indeterminacy component is one such point of variation. Some definitions use $I_A^c = 1 - I_A$, while others preserve or transform indeterminacy differently. Therefore every theorem must specify the convention being used.

Another caution concerns metrizability. A neutrosophic metric is not a single universally standardized object. Some papers define metric values through neutrosophic numbers, while others use triples of functions or fuzzy-metric-style conditions. This paper therefore states metrizability results in a conservative form, emphasizing induced gauge metrics and compatible graded bases. This avoids claiming a universal metrization theorem that may depend on the chosen definition of neutrosophic metric.

A final caution concerns examples. Many finite examples in neutrosophic topology are sensitive to the selected membership convention. The examples included in this paper are intended to

demonstrate structural phenomena such as failure of converses, connectedness without separation and discreteness implying total disconnectedness. Before using them in a formal submission, the author should check the examples against the exact definition adopted by the target journal.

15. Main Results Summary

Result	Statement	Significance
Theorem 5.4	$N-T_2 \Rightarrow N-T_1 \Rightarrow N-T_0$	Establishes basic separation hierarchy
Theorem 5.8	$N-T_4 \Rightarrow N-T_3$	Links normality and regularity
Theorem 5.9	$N-T_3 \Rightarrow N-T_2$	Shows regular T_1 spaces are Hausdorff
Proposition 6.1	$N-T_i$ is hereditary to subspaces for $i = 0,1,2$	Supports subspace theory
Theorem 7.2	Connected iff no open separation	Characterizes connectedness
Theorem 7.4	Continuous image of connected is connected	Preserves structure under maps
Theorem 8.3	Continuous image of compact is compact	Extends compactness theory
Theorem 9.4	Neutrosophic metric topology is $N-T_2$	Metric spaces are Hausdorff
Theorem 9.5	Neutrosophic metric topology is first countable	Connects metrics and local bases
Theorem 9.6	Lifted metrizable shadow implies neutrosophic metrizable	Provides a usable metrizable criterion

16. Discussion

The results show that neutrosophic topology retains much of the logical pattern of classical topology while introducing additional degrees of freedom through indeterminacy. The hierarchy $N-T_4 \Rightarrow N-T_3 \Rightarrow N-T_2 \Rightarrow N-T_1 \Rightarrow N-T_0$ parallels the classical hierarchy. Nevertheless, the examples show that the converses fail in the expected ways.

Connectedness behaves similarly to classical connectedness when expressed through clopen sets. However, the interpretation of a clopen set in the neutrosophic context is more refined because truth, indeterminacy and falsity values interact. Strong connectedness and super-connectedness represent genuinely neutrosophic refinements of ordinary connectedness.

Compactness also follows classical patterns when stated via open covers. Its interaction with Hausdorffness, closed subspaces and continuous functions remains stable under the neutrosophic formalism. This makes compactness an important bridge between classical and neutrosophic topological arguments.

Metrizability is more delicate. A neutrosophic metric need not be merely a crisp metric in disguise; it may contain an indeterminacy component. Nevertheless, a useful practical metrizable criterion is available when the crisp shadow is metrizable and the neutrosophic

topology is generated by compatible graded lifts of a metric base. This approach avoids overclaiming while providing a workable construction.

17. Conclusion

This paper has presented a publication-style investigation of separation axioms, connectedness, compactness and metrizability in neutrosophic topological spaces. Definitions were organized consistently, the main implications among separation axioms were proved, connectedness was characterized through clopen sets, compactness results were recorded and sufficient conditions for metrizability were developed.

The study confirms that neutrosophic topology is not merely a cosmetic extension of fuzzy topology. Its explicit treatment of indeterminacy enriches the behaviour of open sets, closed sets and continuous mappings. At the same time, many classical topological results remain valid when reformulated carefully.

Future work may proceed in several directions. First, stronger metrization theorems for neutrosophic spaces should be developed. Second, paracompactness, local compactness and uniform structures in neutrosophic topology deserve systematic treatment. Third, interactions between neutrosophic topology and algebraic structures such as neutrosophic groups, rings and graphs may yield useful applications. Finally, applied work may investigate how neutrosophic topological models can support decision systems involving incomplete, inconsistent or indeterminate data.

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