Higher Dimensional Generalization of Tolman's Schwarzschild Interior Solution in General Relativity

Dr. Ramesh Kumar
(Research Scholar)
Registration No-XM/R/285/2022
Department of Mathematics
Magadh University Bodh-Gaya (Bihar)-824234, India.

ABSTRACT

Here in this interesting paper is dedicated to find and generalize the famous Richard C. Tolman's Schwarzschild interior solution in higher dimensions. This solution is very important and useful in understanding and discussing the internal construction of stars. Max Wyman's generalization has been also discussed to some extent. Energy density ρ and pressure p have been found. These solutions can be matched at the boundary with the exterior solution (Myers and Perry, 1986).

$$ds^2 = \left[1 - \frac{W}{r^{D-3}}\right] dt^2 - \left[1 - \frac{W}{r^{D-3}}\right]^{-1} dr^2$$

$$-r^2(d\theta_1^2 + \sin^2\theta_1\, d\theta_2^2 + \ldots.. + \sin^2\theta_1 \ldots. \sin^2\theta_n\, d\theta_{n+1}^2)$$

where W related to total mass of the fluid inside a sphere of radius $\mathbf{r_b}$

Key words: dimension, stars, energy, density, Pressure, universe, field-equation

1. INTRODUCTION

Einstein's field equations which connect the distribution of matter and energy with the gravitational field (described by the potential g_{ij}) are

$$-8\pi T_{ij} = R_{ij} - \frac{1}{2} R g_{ij} + \wedge g_{ij}$$

where T_{ij} is the material energy momentum tensor, R_{ij}. The Ricci tensor, R, the curvature invariant (or scalar curvature tensor) and $\wedge$, the cosmological constant. The field equations being highly non-linear, the exact solutions are obtained only in a few special cases. Professor Richard C. Tolman [27] considered the problem which corresponds to an equilibrium distribution of perfect fluid and obtained solutions under some mathematical restrictions. Here we discuss and obtain higher dimensional generalization of Tolman's (1939) solution which are significant in the study of the internal structure of stars.

If we set $\wedge = 0$ in agreement with known fact that the cosmological constant is too small to produce appreciable effective within a moderate spatial range. The above Einstein's field equations (with $\wedge$=0) may be written as

$$-8\pi T_{ij} = R_{ij} - \frac{1}{2} R g_{ij}$$

Where velocity of light and gravitational constant is taken to be unity in the usual units.

The explanation of the smallness of the extra dimensions of the universe by the dynamical evolution of the latter has also been proposed in the case of a more realistic model (11-dimensional supergravity). Now the investigations in higher dimensions have been important in view of recent developments of superstring theory in which the space-time is considered to be of dimension higher than four. The generalization of solutions of Einstein's field equations to higher dimensions thus has become necessary. Higher dimensional extension of Schwarzschild, Reissner – Nordstrom and Kerr solutions have been provided by Myers and Perry [20]. The Reissner-Nordstrom-de Sitter metric and Kerr-de Sitter metric has been obtained by Dianyan for higher dimensional space-time. Higher dimensional generalization of Schwarzschild interior, Floride's [8] and Marder's [18] solutions has been obtained by Krori *et. al.* [12, 13, 14], Shen and Tan [22, 23] have obtained higher dimensional generalization of interior Wyman's ($\rho = \mu r^a$) solution [29] and global regular solution with equation of statue p + ρ = 0. The generalization of Mehra's [19], Whittaker's [28], Ibrahim-Nutku's [9] solutions; solutions for superdense, disordered radiation, constant gravitational mass density; Adler [3], Kuchowicz's [15, 16] and Bayin's [4] solutions to higher dimensional space-time has been presented by Singh *et. al*. [25]. Some other workers in this field are Koikawa [11], Iyer and Vishveshware [10], Liddle *et. al*. [17], Chatterjee *et. al*. [5, 6], Singh and Yadav [26] Singh and Kumar [24] and Ahmed and Alamri [2].

Here in the chapter, we find and generalize the famous Richard C. Tolman's [27] solutions in higher Dimensional Generalization of Tolman's Schwarzschild Interior Solution Energy density ρ and pressure p have been found. These solutions can be matched at the boundary with the exterior solution (Myers and Perry [20]).

2. THE FIELD EQUATIONS

We take the static spherically symmetric metric for D = n+3-dimensional space-time in the form given by (Myers and Perry [20].

(2.1) $$ds^2 = e^B dt^2 - e^A dr^2 - r^2(d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + ...$$

$$.....+\sin^2\theta_1 \sin^2\theta_2 ....\sin^2\theta_n d\theta_{n+1}^2)$$

where A = A(r) and B = B (r). Also

$$x^0 = t, x^1 = r, x^2 = \theta_1, x^3 = \theta_3 \text{ etc.}$$

The Einstein field equations for D = (n + 3) dimensional space. time are

$$(2.2)\quad R_{ij} = -8\pi G(T_{ij} - \frac{1}{D-2} g_{ij} T_k^k)$$

The energy momentum tensor for matter is given by

$$(2.3)\quad T_{ij} = \text{diag}(\rho, \underbrace{-p, \ldots\ldots -p)}_{n+2}$$

where ρ is the material density and p, are the pressure.

In usual units, we take velocity of Hight c = 1 and gravitational constant G = 1. The field equations (2.2) for the metric (2.1) yield

$$(2.4)\quad e^{-A}\left(\frac{A'}{r} - \frac{n}{r^2}\right) + \frac{n}{r^2} = \frac{16\pi}{(n+1)}\rho$$

$$(2.5)\quad e^{-A}\left(\frac{B'}{r} + \frac{n}{r^2}\right) - \frac{n}{r^2} = \frac{16\pi}{(n+1)}p$$

$$(2.6)\quad e^{-A}\left(\frac{B''}{2} + \frac{B'^2}{4} - \frac{A'B'}{4} - \frac{B' + nA'}{2r} - \frac{n}{r^2}\right) + \frac{n}{r^2} = 0$$

$$(2.7)\quad p' + (p+\rho)\frac{B'}{2} = 0$$

We multiply equation (2.6) by 2/r and then arranging the terms, we obtain

$$(2.8)\quad \frac{d}{dr}\left[\frac{n(e^{-A}-1)}{r^2}\right]+\frac{d}{dr}\left[\frac{e^{-A}B'}{2r}\right]+e^{-A-B}\frac{d}{dr}\left[\frac{e^{B}B'}{2r}\right]=0$$

3. SOLUTIONS OF THE FIELD EQUATION

The general solution of equation (2.8) as such is difficult to obtain, therefore to make the equation (2.8) integrable, we use some judicious conditions on A or B or relation between A and B and then give resulting solution for e^A, e^B, ρ and p as functions of r which can be obtained by combining the new equation with (2.8), (2.4) and (2.5). Below we discuss the specific solution obtained by Tolman in higher dimensions

4. EINSTEIN UNIVERSE IN HIGHER DIMENSIONS

Here we choose

(4.1) e^B = constant = k (say)

With this assumption equation (2.8) becomes immediately integrable because the second two terms vanish due to the constancy of e^B and we get

$$(4.2)\quad \frac{d}{dr}\left[\frac{n(e^{-A}-1)}{r^2}\right]=0$$

which on integration yields

$$\frac{n(e^{-A}-1)}{r^2}=\text{constant}=-\frac{1}{R^2}\,(\text{say})$$

which gives

$$(4.3)\quad e^{-\lambda}=1-\frac{r^2}{nR^2}$$

Use of equation (4.1) in equations (2.5) and (4.3), we find pressure p as

$$(4.4)\quad p = \frac{(n+1)}{16\pi R^2}$$

Similarly, from equations (2.4) and (4.3), we find density ρ as

$$(4.5)\quad \rho = \frac{(n+2)(n+1)}{16\pi n R^2}$$

The resulting solution has found popularity in static cosmology. It is the analogue of static Einstein universe in higher dimension with uniform pressure and density.

5. SCHWARZCHILD INTERIOR SOLUTION

In this case-l

we assume

$$(5.1)\quad e^{-A} = \left[1 - \frac{r^2}{nR^2}\right]$$

which on substitution in (2.8) simplifies it since first term vanishes and we set

$$\frac{d}{dr}\left[\frac{e^{-A}B'}{2r}\right] + e^{-A-B}\frac{d}{dr}\left[\frac{e^{B}B'}{2r}\right] = 0$$

which on integration finally provides the solution

$$(5.2)\quad e^{B} = \left[\alpha - nkR^2\left(1 - \frac{r^2}{nR^2}\right)^{1/2}\right]^2$$

where α and k are constants Now using equation (5.1) and (5.2) in (2.4) and (2.5) the pressure and density are found to be

$$(5.3)\quad p = \frac{(n+1)}{16\pi R^2}\left[(n+2)kR^2\left(1 - \frac{r^2}{nR^2}\right)^{1/2} - \alpha\right]$$

$$\left[\alpha - nkR^2\left(1-\frac{r^2}{nR^2}\right)^{1/2}\right]^{-1}$$

(5.4) $\rho = \dfrac{(n+1)(n+2)}{16\pi nR^2}$

The solution can be considered as higher dimensional analogue of well-known Schwarzschild interior solution for a fluid sphere of constant density. When k = 0, the solution provides higher dimensional analogue of Einstein universe as obtained in

Case-II

When $\alpha = 0$, we find

(5.5) $e^{-A} = \left(1-\dfrac{r^2}{nR^2}\right)$ and $e^B = \text{constant}\left(1-\dfrac{r^2}{nR^2}\right)$

which is higher dimensional analogue of de-Sitter universe.

Case - III

In this case we take

(5.6) $e^B \dfrac{B'}{2r} = \text{constant}$

With this, the third term of equation (2.8) vanishes and we get

(5.7) $\dfrac{d}{dr}\left(\dfrac{e^{-A}-1}{r^2}\right) + \dfrac{d}{dr}\left[\dfrac{e^{-A}B'}{2r}\right] = 0$

$$\Rightarrow \quad n\frac{(e^{-A}-1)}{r^2} + \frac{e^{-A}B'}{2r} = \text{(constant)}$$

Integrating (5.6) we get

(5.8) $e^B = \mu^2\left[1+\dfrac{r^2}{\lambda^2}\right]$ (λ, μ are constants)

Which on combination with (5.7) finally gives

(5.9) $$e^{A} = \left[\frac{1 + \frac{(n+1)r^2}{n\lambda^2}}{\left(r - \frac{r^2}{nR^2}\right)\left(1 + \frac{r^2}{\lambda^2}\right)} \right]$$

On substituting e^{-A}, A' and B' from (5.8) and (5.9) we can easily find p and ρ from equations (2.4) and (2.5) as

(5.10) $$\frac{16\pi}{(n+1)} p = \frac{1}{\lambda^2} \left[\frac{1 - \frac{\lambda^2}{R^2} - \frac{(n+2)r^2}{nR^2}}{\left[1 + \frac{(n+1)r^2}{n\lambda^2}\right]} \right]$$

(5.11) $$\frac{16\pi}{(n+1)} \rho = \frac{1}{\lambda^2} \left[\frac{1 + \frac{(n+2)\lambda^2}{nR^2} + \frac{(n+2)r^2}{nR^2}}{\left[1 + \frac{(1+n)r^2}{n\lambda^2}\right]} \right] + \frac{2}{\lambda^2} \left[\frac{1 - \frac{r^2}{nR^2}}{\left[1 + \frac{(1+n)r^2}{n\lambda^2}\right]^2} \right]$$

The line element describing this solution can be written using e^A and e^B in equation (2.1)

At the centre of sphere, the pressure (p_C) and density (ρ_C) can be found by putting r = 0 in equations (5.10) and (5.11) and they are

(5.12) $$p_C = \left(\frac{1}{A^2} - \frac{1}{R^2} \right) \left(\frac{(n+1)}{16\pi} \right)$$

and

(5.13) $$\rho_C = \left(1 + \frac{2}{n}\right) \left(\frac{1}{A^2} + \frac{1}{R^2} \right) \left(\frac{(n+1)}{16\pi} \right)$$

The equations (5.10) – (5.13) can be combined in the convent simple form given by

(5.14) $$\rho = \rho_C - (n+4)(p_C - p) - \frac{4(n+1)(p_C - p)^2}{n(\rho_C + p_C)}$$

Which is known as equation of state connecting the density and pressure of the fluid inside the sphere.

At the boundary of the sphere the pressure drops to zero and the boundary density ρ_b has the value for equation (5.14)

(5.15) $\rho_b = \rho_C - (n+4)p_C - \dfrac{4(n+1)p_c^2}{n(\rho_C + p_C)}$

From equation (5.10) we find that at the boundary $r = r_b$ of sphere (where p = 0), we have

(5.16) $r_b = R\left[\dfrac{n}{(n+2)}\left(1-\dfrac{\lambda^2}{R^2}\right)\right]^{1/2}$

It is clear that with $R^2 > \lambda^2$, the pressure and density of the fluid fall from their central to their boundary values where the density still remains positive.

The solution may be a useful one in studying properties of spheres of compressible fluid in higher dimensions since the equation of state (5.14) is relatively simple.

These solutions can be matched at the boundary $r = r_b$ with the exterior solution (Myers and Perry [20])

(5.17) $(ds^2 = \left[1-\dfrac{W}{r^{D-3}}\right]dt^2 - \left[1-\dfrac{W}{r^{D-3}}\right]^{-1} dr^2$

$$-r^2\left(d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + ... + \sin^2\theta_1 ....\sin^2\theta_n d\theta_{n+1}^2\right)$$

where w related to total mass of the fluid inside a sphere of radius r_b given by

(5.18) $M = \dfrac{1}{2}W(D-2)C_{D-2}$

were

(5.19) $C_{D-2} = 2\pi^{(D-2)/2}\sqrt{(D-1)/2}$

REFERENCES

1. Accetta, F. and Gleiser, M. (1987). Ann. Phys. (N.Y) 176, 278.
2. Ahmed, N and Alamri, S.Z. (2018), Res Astron, Astro phys, 18, 123.
3. Adler, R.J. (1974), J. Math. Phys., 15, 727.
4. Bayin, S.S. (1978), Phys. Rev. D, 18, 2745.
5. Chatterjee, S., Banerjee, A. And Bhui, B. (1990a), Phys. Lett., A, 149, 91.
6. Chatterjee, S., Bhui, B. and Banerjee, A. (1990b). J. Math. Phys., 31, 2203.
7. Dianyan, Xu (1988); Class Quant. Grav. (U.K.), 5, 871
8. Florides, P.S. (1974), Proc. Roy. Soc., A, 337, 592.

9. Ibrahim, A. and Nutku, Y. (1970), Gen. Rel. Grav., 7, 949.

10. Iyer, B. R. and Vishveshwara, C.V. (1989), Pramana (India), 32, 749.

11. Koikawa, T. (1986), Phys. Lett. A. 117, 279.

12. Krori, K.D., Borgohain, P. and Das, K. (1988), Phys. Lett. A, 132, 321.

13. Krori, K.D., Borgohain, P. and Das, K. (1989), Gen. Rel. Grav., 21, 1099.

14. Krori, D.K., Borgohain, P. and Das, K. (1990), J. Math. Phsy., 31, 149.

15. Kuchowicz, B. (1968), Acta, Phys. Polon B, 33, 541.

16. Kuchowicz, B. (1975), Astrophys. Space Sci., 33, L13.

17. Liddle, A.R., Moorhouse, R.G. and Henrigues, A.B. (1990), Class Quant. Grav. (U.K.), 7, 1009.

18. Marder, L. (1958), Proc. R. Soc. London Ser. A. 244, 524.

19. Mehra, A.L. (1966), J. Austr. Math. Soc., 6, 153.

20. Myers, R.C. and Perry, M.J. (1986), Ann. Phys. (NY), 172, 394

21. Sahdev, D. (1984), Phys. Rev., D, 30, 2495.

22. Shen, Y.G. and Tan, Z.Q. (1988a), Phys. Lett. A, 137, 96.

23. shen, Y.G. and Tan, Z.Q. (1989b), Phys. Lett. A. 142, 341.

24. Singh, S. and Kumar, M. (2019), Res. Rev. J., 4, 1684-88.

25. Singh, T. *et. al.* (1992), Int. Journ. Theo. Phys., 31, 545.

25. (a) Singh, T. *et. al.* (1993), Astro Phys. Space Sci., 199, 113.

26. Singh, T. and Yadav, R.B.S. (1981), J. Math Phys. Sci. 15, 283.

27. Tolman, R.C. (1939), Phys. Rev., 55, 364.

28. Whittaker, J.M. (1968), Proc. Roy, Soc. A306, 1.

29. Wyman, M. (1949). Phys. Rev., 75, 1930.