Spherically Symmetric Radiating Universe in General Relativity

Dr. Sujit Kumar
Research Scholar
University Department of Mathematics
M.U. Bodh-Gaya (Bihar)

ABSTRACT :

The present paper provides an exact static spherically symmetric solution of Einstein's field equations using the equation of state $\rho = 3p$ and also with judicious choice of metric potential g_{44}. We have also found and discussed various physical and geometrical properties of the model.

Key Words: Static, spherical symmetry, equation of state, metric, potential.

1. Introduction

Various authors have focussed on study of a state in which radiation is concerned in general relativity. A static cylindrically symmetric perfect fluid solution describing disordered radiation having $p = 1/3\rho$ was obtained by Teixeira. Wolk and Som [21] and Kramer [8]. In these solutions the cylinder of fluid was radially infinite and the fluid possessed finite pressure and density everywhere, decreasing monotonically to zero outwards. The g_{ij} in these static cases involved simple algebraic functions only.

The system of an electromagnetic radiation involving only under involving only under influence of its own gravitation and pressure effects has been one of the most fascinating physical system described by general relativity. In this line Klein [7] obtained an approximate solution of Einstein's equation for a distribution of diffused radiation with spherical symmetry, which he presented as a set of series expansions. This distribution in equilibrium shows maximum condensation at the centre and dilutes monotonically to a zero value at infinity. However his solution at infinity does not coincide with the vacuum solution of Schwarzschild [17]. Now stationary inhomogeneous solutions to Einstein's equation for an irrolational perfect fluid have featured equations of state $p = p$ [12, 13, 19, 24], $p = \rho + \text{const.}$ [1]. $P = \lambda\rho$, (λ =

const.) [23] and p = 1/3 ρ [6, 8]. The solutions with equation of state p = 3ρ obtained by Feinstein and Senovilla [6] is not the same as that for the case λ = 1/3 derived by Wainwright and Goode [23] although in both solutions g_{ij} depends on simple hyperbolic functions of a space co-ordinate and a time co-ordinate. Again the solutions have p = 1/3ρ given recently by Feinstein and Senovilla [6] is dinstinct from the previous solutions and depends only on hyperbolic functions.

The general relativity finds in interesting application to an investigation of state in which radiation is concentrated around a star. Raj Bail and Jain [16] have obtained magnetostatic models filled with dust and disordered radiation in which the distribution is that of perfect fluids [10]. Singh and Abdussattar [18] and Purushottam and Yadav [15(a)]. Obtained an exact solution of Einstein’s field equations for a homogeneous perfect fluid core surrounded by a frozen photon field. Teixeira et. al. [20] obtained an exact solution of an unbounded plane symmetric distribution of disordered radiation.

Similar to Klein’s sphere their slab distribution shows a larger condensation in the innermost regions and dilutes monotonically to a vanishing distribution Outwards, tending acsymptotically to the plane vacuum solution of Levi – Vivita [11]. They have also obtained an exact solution for a distribution of disordered radiation with cylindrical symmetry in equilibrium (1977]. Davidson [2] has presented a solution that provides a non-stationary analog to the static case when p = 1/3ρ, again depending only on algebraic functions of the space co-ordinate r and time co-ordinate t. It is interpreted as an expanding perfect fluid cylinder of infinite radius. The solution can be described as cosmological in the sense that it starts from big-bang infinites but is subsequently well behaved everywhere. In particular, for t > 0 both p and ρ are positive and finite while monotonically decreasing to zero when either r or increase to infinity. This solution is not contained in any of previous solutions. In view of the still uncertain origin of our universe leading to its present high degree of homogeneity and isotropy, it seems worthwhile to confirm that general relativity contains cylindrically symmetric solution that starts from big bang conditions and evolve globally in physically reasonable manner. Such inchonomogeneous solutions extend the possibilities for characterization of the universe in the neighbourhood of the big bang even.

Here in the chapter we have obtained an exact, static spherically symmetric solution of Einstein's field equations using the equation of state ρ = 3p and also with a suitable choice of metric potential e^{ν}. We have also found various physical and geometrical properties of the model.

2. The Field Equations and Their Solutions

We take the metric in the form

(2.1) $$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2$$

where ν and λ are functions of r only the field equations.

(2.2) $$R_j^i \frac{1}{2} R\delta_j^i = -8\pi T_j^i$$

For the metric (2.1) and (Tolman [21])

(2.3) $$-8\pi T_1^1 = e^{-\lambda}\left(\frac{\nu^1}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2}$$

(2.4) $$-8\pi T_2^2 = -8\pi T_3^3 = e^{-\lambda}\left(\frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \frac{\nu'^2}{4} + \frac{\nu' - \lambda'}{2r}\right)$$

(2.5) $$8\pi T_4^4 = e^{-\lambda}\left(\frac{\nu}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2}$$

Where a prime denotes differentiation with respect to r. throughout the investigation we set velocity of light c and gravitational constant k to be unity. A disordered distribution of radiation can be regarded as a perfect fluid having the energy momentum tensor.

(2.6) $$T_1^1 = (\rho + p)u^i u_j - \delta_j^i p$$

characterized by the equation of state

(2.7) $$\rho = 3p$$

We use commoving co-ordinates so that

$$U^1 = u^2 = u^3 = 0 \text{ and } u^4 = e^{-\nu/2}$$

The non-vanishing components of the energy momentum tensor are

$$T_1^1 = T_2^2 = T_3^3 = p \text{ and } T_4^4 = \rho$$

We can then write and field equation

(2.8) $$8\pi p = e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2}$$

(2.9) $$8\pi p = e^{-\lambda}\left(\frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \frac{\nu'^2}{4} + \frac{\nu' - \lambda'}{2r}\right)$$

(2.10) $$8\pi\rho = e^{-\lambda}\left(\frac{\nu'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2}$$

Using equation (2.7), (2.8) and (2.10) we have

(2.11) $$3e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2}$$

From (2.11) we see that if ν is known, λ can be obtained.

So we choose

(2.12) $$e^{\nu} = Dr^2$$

Where D is constant.

Equation (2.12) reduces (2.11) to the for

(2.13) $$3e^{-\lambda}\left(\frac{2}{r^2} + \frac{1}{r^2}\right) - \frac{3}{r^2} = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2}$$

which may be further reduced to

(2.14) $$e_{\lambda}^{-\lambda} r - 10e^{-\lambda} + 4 = 0$$

We put $y = e^{-\lambda}$ so that the equation (2.13) is changed into the form.

(2.15) $$\frac{dy}{dr}+\frac{10y}{r}=\frac{4}{r}$$

Which is a linear differential equation whose solution is

(2.16) $$y=\frac{2}{5}+\frac{c}{r^{10}}$$

Therefore we get

(2.17) $$e^{-\lambda}=\frac{2}{5}+\frac{c}{r^{10}}$$

Where c is constant

Consequently the metric (2.1) can be put into the form.

(2.18) $$ds^2=Dr^2dt^2-\left(\frac{2}{5}+\frac{c}{r^{10}}\right)^{-1}dr^2-r^2(d\theta^2+\sin^2\theta d\phi^2)$$

Absorbing the constant D in co-ordinate differential at the metric (2.18) goes to the form.

(2.19) $$ds^2=r^2dt^2-\left(\frac{2}{5}+\frac{c}{r^{10}}\right)^{-1}dr^2-r^2(d\theta^2+\sin^2\theta d\phi^2)$$

The non vanishing component of Riemann-Christoffel Curvature tensor R_{hijk} for the metric (2.19) are

(2.20) $$R_{1212}=\frac{100c}{2r^{10}+5g}$$

$$R_{2424}=\frac{5r^{12}}{(2r^{10}+5c)}$$

$$R_{1313}=\frac{25\sin^2\theta}{2r^{10}+c}$$

$$R_{1414}=\frac{25c}{r(2r^{10}+5c)}$$

$$R_{3434} = -\frac{5\sin^2\theta r^2}{(2r^{10}+5c)}$$

$$R_{2323} = \frac{5r^{12}\sin^2\theta}{2r^{10+5c}}$$

Choosing the orthonormal tetrad $\bar{\lambda}_i^{\ell}$ as

(2.21) $$\bar{\lambda}1_{(1)} = \left(\frac{5r^{10}}{2r^{10+5c}}\right)^{1/2}(0,0,0)$$

$$\bar{\lambda}1_{(2)} = \left(0,\frac{1}{r},0,0\right)$$

$$\bar{\lambda}1_{(3)} = \left(0,\frac{1}{r\sin\theta},0\right)$$

$$\bar{\lambda}1_{(4)} = \left(0,0,0\frac{1}{r}\right)$$

The physical components $R_{(abcd)}$ of the curvature tensor defined by

(2.22) $$R_{(abcd)} = \bar{\lambda}^i_{(a)}\bar{\lambda}^j_{(b)}\bar{\lambda}^k_{(c)}\bar{\lambda}^{\ell}_{(d)}R_{ijk\ell}$$

are

$$R_{(1212)} = \frac{500Cr^8}{(2r^{10}+5c)^2}$$

$$R_{(2424)} = \frac{5r^5}{2(2r^{10}+5c)}$$

$$R_{(3131)} = \frac{125r^8}{(2r^{10}+5c)^2}$$

$$R_{(1414)} = \frac{125cr^7}{2(2r^{10}+5c)^2}$$

$$R_{(3434)} = \frac{5}{2r^4(2r^{10}+5c)}$$

$$R_{(2323)} = \frac{2r^8}{2r^{10}+5c}$$

We see that $R_{(abcd)} \to 0$ as $r \to \infty$. If follows that the space time is asymptotically homoloidal.

Also for the metric (2.19) the fluid velocity u^i is given by

(2.23) $u^1 = u^2 = u^3 = u_1 = u_3 = 0$

and $u^4 = \frac{1}{r}, u_4 = r$

The scalar of expansion = u^i; I is

identically zero. The non vanishing components of the tensor of rotation w_{ij} defined by

(2.24) w_{ij} = ui, j – u_j , i

are

(2.25) $w_{14} = -w_{41} = -1$

The components of the shear tensor σ_{ij} defined by

(2.26) $\sigma_{ij} = \frac{1}{2}(u_i j + u_{j,i})\frac{1}{3}\theta h_{ij}$

with the projection tensor

$$h_{ij} = g_{ij} - u_1 u_j$$

are

(2.27) $\delta_{14} = \sigma_{41} = -\frac{2}{5}$

The other components being zero.

3. Solutions the perfect fluid Core

Pressure and density for metric (2.19) are

(3.1) $$8\pi p = \frac{8\pi\rho}{3} = \frac{r^{10} + 15c}{5r^{12}}$$

(3.2) $$R^2 = \frac{r_0^2}{\left(\frac{3}{5} - \frac{c}{r_0^{10}}\right)}$$

$$A = \frac{r_0^2 + R^2\left(1 - \frac{r_0^2}{R^2}\right)}{r_o}$$

$$B = \frac{R^2}{r_0}\left(1 - \frac{r_0^2}{R^2}\right)^{1/2}$$

$$C = r_0^{10}\left(\frac{3}{5} - \frac{r_0^2}{R^2}\right)$$

and the density of core

(3.3) $$\rho_0 = \frac{3\left(\frac{3}{5}\frac{r}{r_0^{10}}\right)}{\left(\frac{5\, r_0^{10}}{8\pi, r_0^2}\right)}$$

which complete the solution for the perfect fluid core of radius r_0 surroundes by the fluid with $\rho = 3p$.

4. Discussion

Here we have obtained exact solution for static spherically symmetric solution using equation of state $\rho = 3p$ (disordered radiation). We have also given solution for the perfect fluid core. Such type of investigations where radiation is concerned around a star is much useful and interesting is general relativity.

5. References

1. Allrutt, J.A. (1980); in exact solution of Einstein's Field equations (Cambridge U.P. (Cambridge), P. 227.

2. Davidson, W. (1991); J. Math, Phys., 32, 1560-61.

3. Durgapal, M.C. and Gehlot, G.L. (1968); Phys. Rev., 172, 1308.

4. Durgapal, M.C. and Gehlot, G.L. (1969); Phys. Rev., 183, 1102.

5. Durgapal, M.C. and Gehlot, G.L. (1971); Phys. Rev., D(4), 2963.

6. Feinstein, A. and Senovilla, J.M.M. (1989); Class. Quantum Gravit, 6, L. 89.

7. Klein, O. (1948), Ark. Mat. Astr. Fys. T. 34A, No. 19, 1-11.

8. Krori, K.D. (1970); Indian J. Pure Appl. Phys., 8, 588.

9. Leibivitz, C. (1971): Phys. Rev., 185, 1664.

10. Leibivitz, C. (1972); J. Phys. A., 5, 211.

11. Levicivita, T. (1918); Rend Reale Accad. Lincel, t. 27, 20 som, 240-248.

12. Leteiler, P.S. (1975); J. Math. Phys., 16, 1488.

13. Leteiler, P.S. and Tabensky, R.R. (1975); Nuovo Cimento, B28, 407.

14. Mehra, A.L., Vaidya, P.C. and Kushwaha, R.S. (1973); Gravitation Chap. 23, (Freeman, San Fransisco).

15. Oppenhelmer, J.R. and Voloff, G.M. (1939); Phys. Rev., 55, 374.

15a. Purushottam and Yadav, R.B.S. (2012), IJERT, ol. 1(a), P. 1-14.

16. Raj Bali and Jain, D.R. (1991); Astrophs. Space Sci., 175, 89.

17. Schwarzschild, K. (1916); Sitz, Press-Akad, Wiss., 189.

18. Singh, K.P. and Abdussattar (1973); Ind J. Pure Appl. Math, 4, 468-472.

19. Singh, T. and Yadav, R.B.S. (1981); Jour. Math, Phys., Sci., 15, 283.

20. Teixeira, A.f. Da F., and Som, M.M. (1977); J.Phys. A., 10, 1679-1696.

21. Teixeira, A.f. da F., 1 del Wolk and Som, M.M. (1977); II Nuovo Cimento B. 41, 387-397.

22. Tolman, R.C. (1966); Relativity, Thermodynamics and Cosmology, Oxford Univ. Press, P. 242.

23. Wainwringht, J. and Goode, S.W. (1980); Phys. Rev. D. 22, 1906.

24. Yadav, R.B.S., Singh, S.S., Prasad, R and Prasad, D. (2003) Proc. Math. Soc., 19, 147.