Assessing the Performance Properties of Pre-test Estimator of Disturbance Variance in Linear Regression Model

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Abstract

This paper explores the performance properties of pre-test estimator of disturbance variance based on criteria such as **bias and risk, under quadratic error loss,** under different conditions**.** Attempt has also been made to carry out a comparative study of Pre-test estimator of disturbance variance with ordinary least squares and restricted least squares estimators.

Keywords: Ordinary Least Squares Estimator (OLS), Restricted Least Squares Estimator (RLS), Pre-Test Estimator, Quadratic Error Loss.

AMS Mathematics Subject Classification (2010) - 62J05

Introduction

In regression analysis, the residuals play an important role. These are not only helpful in the estimation of the parameters of the of the regression model, but also serve as a measure of goodness of fit of the model and thus help the researcher in the postulating the most appropriate regression model. Another important use of the residual sum of squares is to construct the estimators of the disturbance variance in linear regression model when it is unknown. Owing to extensive use of disturbance variance in regression analysis, accurate and improved estimation of disturbance variance is imperative and thus has received enormous attention. Bancroft (1944) explored the idea of the pre-test estimator and studied its performance properties under quadratic error loss function. Such estimator incorporates the sample information with the non-sample information about the value of the parameter. Extensive works have been carried out since then concerning risk properties associated with various estimators of linear model that explicitly utilize prior linear constraints on the model parameters, including restricted least squares and Pre-test estimator [see; Bock et. al. (1973), Judge and Bock (1978)]. In a seminal work Clarke et. al (1987a, b) derive exact finite sample expression for the biases and risk of several common pre-test estimators associated with least squares, maximum likelihood and minimum mean squared error of pre-test estimator in linear regression model. Ohtani (2001) examine the small sample properties of the Pre-test iterative variance estimator and shown that it dominates the iterative variance estimator without pre-testing in term of mean square error with an appropriate critical value. In the field of pre-test estimator for various model parameter Khan et.al. (2005) and Saleh (2006) studied the performance properties of four alternative estimators, including the pre-test estimator of the intercept parameter of simple linear regression model. Their study shows that under certain conditions pre-test estimator dominates the least squares estimator. Kumar (2016) studied the exact distribution and performance properties of Pre-test estimator of regression coefficient under orthonormal regression model.

The focus of this paper, therefore, is to study the performance properties of disturbance variance estimator. In order to study the performance of the Pre-test estimator of disturbance variance, distribution and density function of Pre-test estimator, proposed by Kumar (2020) is used. The plan of this paper is as follows: Section 2 describes the model and estimators. In Section 3 with the help of probability density function, moments of the Pre-test estimators of the coefficients are obtained which eventually lead to the computation of bias and risk of Pre-test estimator. Numerical computation and comparison of estimators are made in section 4. Lastly, in Appendix, proofs of theorems are provided.

2 The Model and The Estimators

Consider the standard multiple linear regression model

$$y = X\beta + \varepsilon \tag{2.1}$$

where y is an $n \times 1$ vector of observations on the response variable, X is an $n \times p$ full column rank nonstochastic matrix of n observations on p explanatory variables, β is a $p \times 1$ vector of unknown parameters associated with the p regressors and ε is an $n \times 1$ vector disturbances. The elements of the disturbances vector ε are assumed to be independently and identically distributed each following normal distribution with mean zero and variance σ^2, so that $E(\varepsilon) = 0$ and $E(\varepsilon\varepsilon') = \sigma^2 I_n$ where σ^2 is finite but unknown. Suppose some prior information in the form of restrictions on β are available and is given by

$$r - R\beta = \delta \tag{2.2}$$

where R is $q \times p$ $(q \leq p)$ matrix of known elements with rank q, q being the number of restrictions imposed on the coefficients, r is a $q \times 1$ vector of known elements and δ is $q \times 1$ vector representing the errors in the restrictions. The least squares estimator without and with restrictions are given by

$$b = (X'X)^{-1}X'y \qquad \text{and} \tag{2.3}$$
$$b_R = (X'X)^{-1}R'[R(X'X)^{-1}R']^{-1}(r - Rb) \tag{2.4}$$

As σ^2 is generally unknown, it has to be estimated from the available sample information. In view of this, the following estimators of disturbance variance, one entirely based on the sample information and the other utilizing prior information may be constructed. Thus using the ordinary least squares estimator (2.3) and restricted least squares estimator (2.4) the following estimators of disturbance variance are constructed

$$s^2 = \frac{1}{m} e'e\,; \qquad m = n - p + \theta \tag{2.5}$$
$$s_R^2 = \frac{1}{m'} e_R' e_R\,; \qquad m' = n - p + q + \theta \tag{2.6}$$

Here e and e_R are the residual vector obtained using ordinary least squares and restricted least squares estimator respectively. *m and m'* are the arbitrary scalars characterizing the estimators.

In order to test the compatibility of the sample information (2.1) and the non-sample information (2.2), the following test of hypothesis may be performed.

$$H_0 : \delta = 0 \tag{2.7}$$

against

$$H_1 : \delta \neq 0 \qquad \text{where } \delta = R\beta - r. \tag{2.8}$$

This hypothesis is tested using the Wald test statistics, given by

$$u = \frac{(r-Rb)'\left[R(X'X)^{-1}R'\right]^{-1}(r-Rb)\big/q}{e'e\big/v} \tag{2.9}$$

The test statistic u has a central F distribution with q and v degrees of freedom and under (2.7) , the test statistics u has a non-central F distribution with q and v degrees of freedom with non-centrality parameter λ. As λ is typically unknown, it is usual to test under the null hypothesis H_0, and so reject the hypothesis if $u > F^{\alpha}_{(q,v)} = c$, where c is the critical value.

In view of this, a Pre-test estimator of error variance may be defined which is given by

$$s^2_{PT} = I_{[0,c)}(u)\, s^2_R + I_{(c,\infty]}(u) s^2 \tag{2.10}$$

where $I(.)$ is an indicator function which is one if u falls in the given interval and zero otherwise.

Following Ohtani (2002), Kumar (2020) obtained the distribution and density function of Pre-test estimator (2.10) which are given as

$$F(\tau) = \sum_{i=0}^{\infty} w_i(\lambda)\, G\left(\frac{v+q}{2} + i,\ \frac{m'\tau}{2\,\sigma^2}\right) \mathrm{I}_{\frac{qc}{v+qc}}\left(\frac{q}{2} + i,\ \frac{v}{2}\right)$$

$$+ \sum_{i=0}^{\infty} w_i(\lambda) \frac{1}{B\left(\frac{q}{2}+i, \frac{v}{2}\right)} \int_{\frac{qc}{v+qc}}^{1} g^{\frac{q}{2}+i-1}(1-g)^{\frac{v}{2}-1}\ G\left(\frac{v+q}{2} + i,\ \frac{m\,\tau}{2\,\sigma^2(1-g)}\right) dg \tag{2.11}$$

where $w_i(\lambda) = \frac{e^{-\lambda}\lambda^i}{i!}$, $G(a,x)$ is the incomplete gamma function and $\mathrm{I}_x(a,b)$ is the incomplete Beta function.

From the above equation (2.11), the density function of the Pre-test estimator may be obtained by differentiating $F(\tau)$ with respect to τ.

$$f(\tau) = \sum_{i=0}^{\infty} w_i(\lambda) \left(\frac{m'}{2\,\sigma^2}\right)^{\frac{v+q}{2}+i} \frac{1}{\Gamma\left(\frac{v+q}{2}+i\right)} . \mathrm{I}_{\frac{qc}{v+qc}}\left(\frac{q}{2} + i,\ \frac{v}{2}\right) \times \tau^{\frac{v+q}{2}+i-1} e^{-\left(\frac{m'}{2\,\sigma^2}\right)\tau}$$

$$+ \sum_{i=0}^{\infty} w_i(\lambda) \frac{1}{\Gamma\left(\frac{v}{2}\right)} \left(\frac{m}{2\,\sigma^2}\right)^{\frac{v}{2}} \left[1 - G\left(\frac{q}{2} + i,\ \frac{m'\tau}{2\,\sigma^2} . \frac{qc}{v}\right)\right] \tau^{\frac{v}{2}-1} e^{-\left(\frac{m}{2\,\sigma^2}\right)\tau} \tag{2.12}$$

3. Properties of Pre-test Estimator of Disturbance Variance

Using simple distributional properties, the bias and risk of s_R^2 is given by

$$B(s_R^2) = \frac{\sigma^2}{m'}(2\lambda - \theta);$$
$$R(s_R^2) = \frac{\sigma^4}{(m')^2}[(2\,\lambda - \theta)^2 + 2(v + q + 4\lambda] \quad \text{where } m' = v + q + \theta$$

The estimator s_R^2 is clearly a biased estimator. Interestingly, this bias does not vanish even if we choose $\theta = 0$. However, if the restrictions are correct, i.e., $\delta = 0$ and so $\lambda = 0$, the situation is much easier to deal. Thus in the situation when $\delta = 0$, s_R^2 is unbiased when $\theta = 0$, consistent when $\theta = p + q$ and a minimum mean squared error estimator when $\theta = 2$. In the same scenario, s_R^2 uniformly dominates s^2 under mean squared error criterion clearly indicating that the prior information, if correct, improves the estimator of error variance

Using the density function (2.12), proposed by Kumar (2020), moments of different order which will eventually lead to determine the bias, mean squared error and risk of Pre-test estimator s_{PT}^2, under quadratic error loss may be obtained. The following theorem gives the raw moments of order j of the Pre-test estimator.

Theorem 1. When errors in the model (2.1) are normally distributed, the j^{th} raw moment of the Pre-test estimator s_{PT}^2 is given by

$$E[s_{PT}^2\,]^j = \left(\frac{2\,\sigma^2}{m'}\right)^j \sum_{i=0}^{\infty} w_i(\lambda)\, \frac{\Gamma\left(\frac{v+q}{2}+i+j\right)}{\Gamma\left(\frac{v+q}{2}+i\right)}.\,\mathrm{I}_{\frac{qc}{v+qc}}\left(\frac{q}{2}+i,\ \frac{v}{2}\right)$$
$$+\left(\frac{2\,\sigma^2}{m'}\right)^j \sum_{i=0}^{\infty} w_i(\lambda)\frac{\Gamma\left(\frac{v}{2}+j\right)}{\Gamma\left(\frac{v}{2}\right)}\ \left[1 - \mathrm{I}_{\frac{qc}{v+qc}}\left(\frac{q}{2}+i,\ \frac{v}{2}+j\right)\right] \qquad (3.1)$$

Proof: See Proof of the Theorems.

Substituting various values of j moments of different orders of the Pre-test estimator of the disturbance variance may be obtained. Thus using Theorem 1, determine the bias and the risk of the estimator s_{PT}^2

$$B(\,s_{PT}^2) = E[s_{PT}^2 - \sigma^2] \qquad (3.2)$$

$$R(\,s_{PT}^2) = E[s_{PT}^2 - \sigma^2]^2 = E[s_{PT}^2]^2 - 2\sigma^2 E[s_{PT}^2] + \sigma^4 \qquad (3.3)$$

The following theorem provides the bias and the risk under quadratic error loss of the Pre-test estimator s_{PT}^2 .

Theorem 2: When errors in the model (2.1) are normally distributed, the bias and the risk of the Pre-test estimator under quadratic error loss is given by

$$B(\,s_{PT}^2) = \frac{2\,\sigma^2}{m'}\sum_{i=0}^{\infty} w_i(\lambda)\ \left(\frac{v+q}{2}+i\right).\,\mathrm{I}_{\frac{qc}{v+qc}}\left(\frac{q}{2}+i,\ \frac{v}{2}\right)$$
$$+\frac{2\,\sigma^2}{m}\sum_{i=0}^{\infty} w_i(\lambda)\ \frac{v}{2}\left[1 - \mathrm{I}_{\frac{qc}{v+qc}}\left(\frac{q}{2}+i,\ \frac{v}{2}+1\right)\right] - \sigma^2 \qquad (3.4)$$

$$R(s_{PT}^2) = 4\sigma^4 \sum_{i=0}^{\infty} w_i(\lambda) \left\{ \frac{1}{m'}\left(\frac{v+q}{2}+i\right)\left[\frac{1}{m'}\left(\frac{v+q}{2}+i+1\right)-1\right] \mathrm{I}_{\frac{qc}{v+qc}}\left(\frac{q}{2}+i,\frac{v}{2}\right) + \frac{1}{m}.\frac{v}{2}\left[\mathrm{I}_{\frac{qc}{v+qc}}\left(\frac{q}{2}+i,\ \frac{v}{2}+1\right) - 1 + \frac{1}{m}\left(\frac{v}{2}+1\right)\left(1-\mathrm{I}_{\frac{qc}{v+qc}}\left(\frac{q}{2}+i,\ \frac{v}{2}+2\right)\right)\right]\right\} + \sigma^4 \qquad (3.5)$$

Proof: See Proof of the Theorems

The expression (3.4) and (3.5) are difficult to interpret. However, it is possible to determine an optimum value of the risk of pre-test estimator. For this purpose, determine the optimum critical value and differentiate (3.5) with respect to c, by using the formula

$$\frac{\partial}{\partial c}.\mathrm{I}_{\frac{qc}{v+qc}}(a,b) = \frac{1}{B(a,b)}.\frac{q^q v^b c^{a-1}}{(v+qc)^{a+b}}$$

Therefore,

$$\frac{\partial}{\partial c}R(s_{PT}^2) = 4\sigma^4 \sum_{i=0}^{\infty} w_i(\lambda) \left\{\frac{1}{m'}\left(\frac{v+q}{2}+i\right)\left\{\frac{1}{m'}\left(\frac{v+q}{2}+i+1\right)-1\right\}\frac{1}{B\left(\frac{q}{2}+i,\frac{v}{2}\right)}.\frac{q^{\frac{q}{2}+i}\, v^{\frac{v}{2}}\, c^{\frac{q}{2}+i-1}}{(v+qc)^{\frac{v+q}{2}+i}} + \frac{1}{m}.\frac{v}{2}\left[\frac{1}{B\left(\frac{q}{2}+i,\frac{v}{2}+1\right)}.\frac{q^{\frac{q}{2}+i}\, v^{\frac{v}{2}+1}\, c^{\frac{q}{2}+i-1}}{(v+qc)^{\frac{v+q}{2}+i+1}} - \frac{1}{m}\left(\frac{v}{2}+1\right)\frac{1}{B\left(\frac{q}{2}+i,\frac{v}{2}+2\right)}\frac{q^{\frac{q}{2}+i}\, v^{\frac{v}{2}+2}\, c^{\frac{q}{2}+i-1}}{(v+qc)^{\frac{v+q}{2}+i+2}}\right]\right\}$$

$$= 4\sigma^4 \sum_{i=0}^{\infty} w_i(\lambda) \frac{\Gamma\left(\frac{v+q}{2}+i+1\right)}{\Gamma\left(\frac{q}{2}+i\right)\Gamma\left(\frac{v}{2}\right)} \frac{q^{\frac{q}{2}+i}\, v^{\frac{v}{2}}\, c^{\frac{q}{2}+i-1}}{(v+qc)^{\frac{v+q}{2}+i}} \left\{\frac{1}{m'}\left\{\frac{1}{m'}\left(\frac{v+q}{2}+i+1\right)-1\right\} + \frac{1}{m}.\frac{v}{2}\left[\frac{\frac{v+q}{2}+i}{\frac{v}{2}}.\frac{v}{(v+qc)} - \frac{1}{m}\frac{\left(\frac{v+q}{2}+i+1\right)\left(\frac{v+q}{2}+i\right)}{\frac{v}{2}}\frac{v^2}{(v+qc)^2}\right]\right\}$$

$$= 4\sigma^4 \sum_{i=0}^{\infty} w_i(\lambda) \frac{\Gamma\left(\frac{v+q}{2}+i+1\right)}{\Gamma\left(\frac{q}{2}+i\right)\Gamma\left(\frac{v}{2}\right)} \frac{q^{\frac{q}{2}+i}\, v^{\frac{v}{2}}\, c^{\frac{q}{2}+i-1}}{(v+qc)^{\frac{v+q}{2}+i}} \left\{\left(\frac{1}{m'} - \frac{1}{m}.\frac{v}{(v+qc)}\right)\left[\left(\frac{v+q}{2}+i+1\right)\left(\frac{1}{m'} + \frac{1}{m}.\frac{v}{(v+qc)}\right)-1\right]\right\}$$

$$\frac{\partial}{\partial c}R(s_{PT}^2) = 4\sigma^4 \sum_{i=0}^{\infty} w_i(\lambda) \frac{\Gamma\left(\frac{v+q}{2}+i+1\right)}{\Gamma\left(\frac{q}{2}+i\right)\Gamma\left(\frac{v}{2}\right)} \frac{q^{\frac{q}{2}+i}\, v^{\frac{v}{2}}\, c^{\frac{q}{2}+i-1}}{(v+qc)^{\frac{v+q}{2}+i}}\,.R_1(c).R_2(c) \qquad (3.6)$$

where $R_1(c) = \frac{1}{m'} - \frac{1}{m}.\frac{v}{(v+qc)} = \frac{m(v+qc)-m'v}{m'm\,(v+qc)}$ (3.7)

and $R_2(c) = \left(\frac{1}{m'} - \frac{1}{m}.\frac{v}{(v+qc)}\right)\left\{\left(\frac{v+q}{2}+i+1\right)\left(\frac{1}{m'} + \frac{1}{m}.\frac{v}{(v+qc)}\right)-1\right\}$ (3.8)

Since $= v+\theta \ \geq 0$ and $m' = v+q+\theta \geq 0$, therefore from equation (3.7) observe that

$$R_1(c) \leq 0 \quad \text{if} \quad c \leq \frac{v}{v+\theta}$$

It is also mention that $i \geq 0$; therefore from (3.8) observe that

$$R_2(c) \geq \left(\frac{v+q}{2} + i + 1\right)\left(\frac{1}{m'} + \frac{1}{m}.\frac{v}{(v+qc)}\right) - 1 \qquad (3.9)$$

Also, if $c \leq \frac{v}{v+2}$, then $qc \leq \frac{qv}{v+\theta}$

$$v + qc \leq \frac{v\,(v+q+\theta)}{v+\theta} \quad \text{and} \quad \frac{v}{v+qc} \geq \frac{v+\theta}{v+q+\theta}$$

therefore, from equation (3.9) it can be easily seen that

$$R_2(c) \geq \frac{v+q+2}{v+q+\theta} - 1 \qquad (3.10)$$

Interestingly, it is easy to see that $R_2(c) = 0$ if $\theta = 2$. This indicates that as c increases from zero to $\frac{v}{v+2}$, the risk of the Pre-test estimator s_{PT}^2 decreases monotonically. Therefore in this paper the properties of the Pre-test estimator s_{PT}^2 in the particular situation when $\theta = 2$ studied. Hence in the case $\theta = 2$, the risk of the Pre-test estimator (3.5) can be written as optimum risk

$$R(s_{PT}^2)_{opt} = 4\sigma^4 \sum_{i=0}^{\infty} w_i(\lambda) \left\{\frac{1}{v+q+2}\left(\frac{v+q}{2} + i\right)\left(\frac{i}{v+q+2} - \frac{1}{2}\right) \mathrm{I}_{\frac{qc}{v+qc}}\left(\frac{q}{2} + i, \frac{v}{2}\right) + \frac{v}{2(v+2)}\left[\mathrm{I}_{\frac{qc}{v+qc}}\left(\frac{q}{2} + i, \frac{v}{2} + 1\right) - \frac{1}{2}.\mathrm{I}_{\frac{qc}{v+qc}}\left(\frac{q}{2} + i,\ \frac{v}{2} + 2\right)\right]\right\} + \sigma^4.\frac{2}{v+2} \qquad (3.11)$$

4. Numerical Computation and Comparison

It is easy to see that when $q = 0\ or\ c = 0$, the risk of Pre-test estimator (3.11) reduce to

$$R(s_{PT}^2)_{q=0} = \sigma^4.\frac{2}{v+2}$$

which is equal to the risk of s^2. It is interesting to note from (3.5) that when the critical value is zero, the bias of the pre-test estimator of disturbance variance is equal to the bias of ordinary least squares estimator of disturbance variance. Consequently, it is unbiased for σ^2 when $\theta = 0$ and critical value is zero. Its risk coincides with the risk under quadratic error loss of ordinary least-squares estimator of disturbance variance in accordance with the theoretical postulations. The numerical computations of the risk of the Pre-test estimator of error variance have been provided in the Tables 1 - 3. It may be noted that the risk values have been computed for $v = 10, 20\ and\ 50$ in order to study the effect of degrees of freedom for change in few chosen values of the non-centrality parameter, λ. The critical value c is also varied in order to get a picture of the behavior of the risk when the size of critical region is varied. As the value of the risk of usual estimator s^2 is the value of the risk at $c = 0$, the pre-test estimator has uniformly smaller risk than the usual estimator when λ is close to zero. As, the value $\lambda = 0$ refers to the case when the prior information is incorporated in the evaluation of the estimator. These values are also presented graphically in Figures 1 to Figure 2. In Figure 1, the risk values look almost coincidental and it is very difficult to analyze the behavior of risk of pre-test estimator of error variance when λ is close to zero. Therefore, Figures 1 (A, B, C) are drawn by selecting few values of λ closer to zero, i.e., $\lambda = .01, .05, .1$; $\lambda = .1, .25, .5$ and lastly away from zero i.e., $\lambda = 1, 5, 10$. Some very interesting results may be seen from plotting the risks by splitting the values of λ as it portrays the relation between the non-centrality parameter and the critical value. When the value of non-centrality parameters is in the range $0 < \lambda < 0.1$, the risk of Pre-test estimator decreases as c increases from 0 to 0.833, has a point of inflection at $c = 0.833$ and again decreases as c exceeds 0.833 (see; Figure 1 (A)). The same decrease in the risk is observed in Figure 1(B) for $\lambda = 0.1, 0.25$, but for $\lambda = 0.5$ the risk of Pre-test estimator decreases as c increases from 0 to 0.833, increases in the interval $0.833 < c \leq 3.0$ and decreases afterwards. Lastly Figure 1 (C) shows that the risk of pre-test estimator also decreases as c increases from 0 to 0.833 but increases afterwards for $\lambda \geq 1$. These results agree with the condition of optimality $c = \frac{v}{v+2}$. and the risk patterns are same even if we increase the degree of freedom.

Table 1. Risk of Pre-test Estimator for selected values of $\sigma^2 = 1, v = 10$, q, λ and c

$v = 10$									
	c	λ =0.01	λ =0.05	λ =0.1	λ =0.25	λ =0.5	λ =1	λ =5	λ =10
$q = 1$	0.0	0.16667	0.16667	0.16667	0.16667	0.16667	0.16667	0.16667	0.16667
	0.1	0.16537	0.16539	0.16541	0.16549	0.16560	0.16579	0.16650	0.16665
	0.5	0.16466	0.16466	0.16467	0.16470	0.16477	0.16491	0.16605	0.16656
	1.0	0.16462	0.16462	0.16462	0.16464	0.16467	0.16479	0.16594	0.16653
	1.5	0.16451	0.16454	0.16457	0.16467	0.16483	0.16515	0.16671	0.16687
	2.0	0.16421	0.16428	0.16436	0.16460	0.16499	0.16572	0.16851	0.16789
	2.5	0.16375	0.16385	0.16399	0.16439	0.16506	0.16633	0.17135	0.16987
	3.0	0.16317	0.16332	0.16351	0.16407	0.16502	0.16688	0.17513	0.17304
	3.5	0.16253	0.16272	0.16295	0.16366	0.16488	0.16733	0.17970	0.17756
	4.0	0.16188	0.16209	0.16236	0.16320	0.16464	0.16765	0.18486	0.18351
	4.5	0.16123	0.16146	0.16176	0.16269	0.16433	0.16785	0.19046	0.19089
	5.0	0.16060	0.16085	0.16117	0.16217	0.16397	0.16793	0.19634	0.19966
$q = 2$	0.0	0.16667	0.16667	0.16667	0.16667	0.16667	0.16667	0.16667	0.16667
	0.1	0.16472	0.16475	0.16479	0.16490	0.16507	0.16536	0.16642	0.16664
	0.5	0.16192	0.16194	0.16196	0.16204	0.16218	0.16250	0.16504	0.16632
	1.0	0.16173	0.16173	0.16174	0.16177	0.16185	0.16208	0.16464	0.16620
	1.5	0.16124	0.16129	0.16136	0.16159	0.16197	0.16276	0.16737	0.16808

	2.0	0.15996	0.16009	0.16025	0.16075	0.16159	0.16332	0.17322	0.17359
	2.5	0.15823	0.15842	0.15866	0.15939	0.16068	0.16341	0.18127	0.18359
	3.0	0.15636	0.15659	0.15688	0.15779	0.15943	0.16302	0.19050	0.19809
	3.5	0.15455	0.15480	0.15512	0.15614	0.15801	0.16228	0.20004	0.21653
	4.0	0.15289	0.15315	0.15348	0.15455	0.15656	0.16132	0.20931	0.23803
	4.5	0.15143	0.15168	0.15201	0.15309	0.15517	0.16026	0.21792	0.26163
	5.0	0.15017	0.15041	0.15073	0.15179	0.15388	0.15917	0.22570	0.28640
$q=5$	0.0	0.16667	0.16667	0.16667	0.16667	0.16667	0.16667	0.16667	0.16667
	0.1	0.16545	0.16547	0.16550	0.16557	0.16567	0.16585	0.16651	0.16665
	0.5	0.15540	0.15545	0.15550	0.15568	0.15598	0.15664	0.16186	0.16523
	1.0	0.15421	0.15422	0.15424	0.15430	0.15444	0.15484	0.15997	0.16445
	1.5	0.15153	0.15164	0.15179	0.15223	0.15301	0.15473	0.16888	0.17645
	2.0	0.14600	0.14619	0.14644	0.14723	0.14866	0.15183	0.18170	0.20340
	2.5	0.14004	0.14027	0.14056	0.14151	0.14326	0.14733	0.19301	0.23884
	3.0	0.13487	0.13510	0.13540	0.13636	0.13819	0.14264	0.20115	0.27588
	3.5	0.13078	0.13099	0.13126	0.13216	0.13393	0.13844	0.20632	0.31018
	4.0	0.12766	0.12784	0.12808	0.12889	0.13054	0.13493	0.20927	0.33985
	4.5	0.12532	0.12547	0.12568	0.12639	0.12791	0.13211	0.21073	0.36449
	5.0	0.12357	0.12369	0.12387	0.12450	0.12589	0.12988	0.21124	0.38446

Figure 1. Risk of Pre-test Estimator for σ^2= $1, v = 10$ and $q = 1$

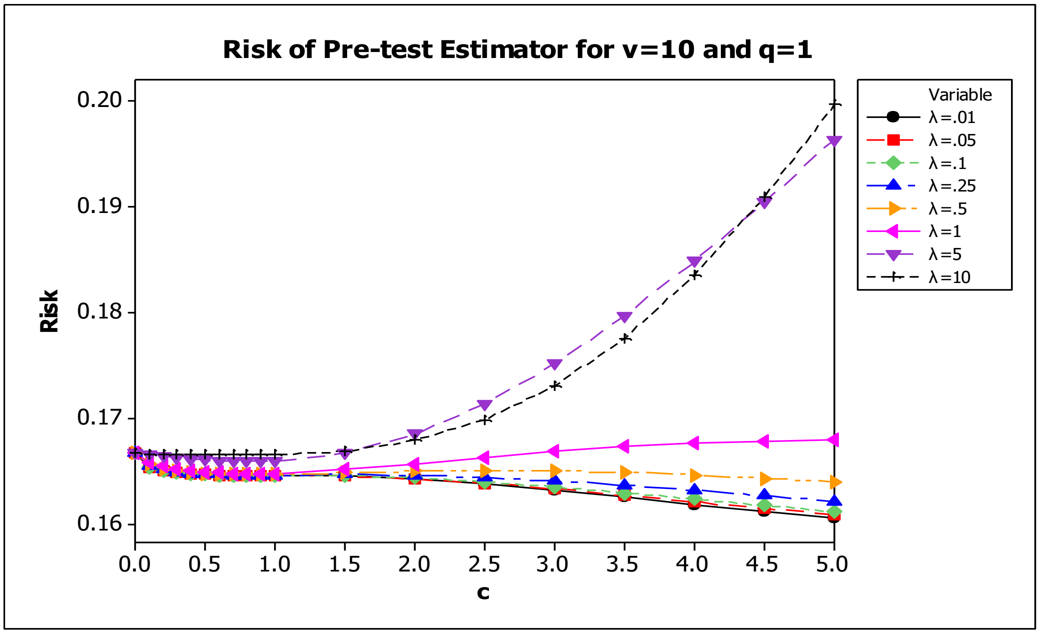


Figure 1(A) Risk of Pre-test Estimator for σ^2= $1, v = 10$ and $q = 1$

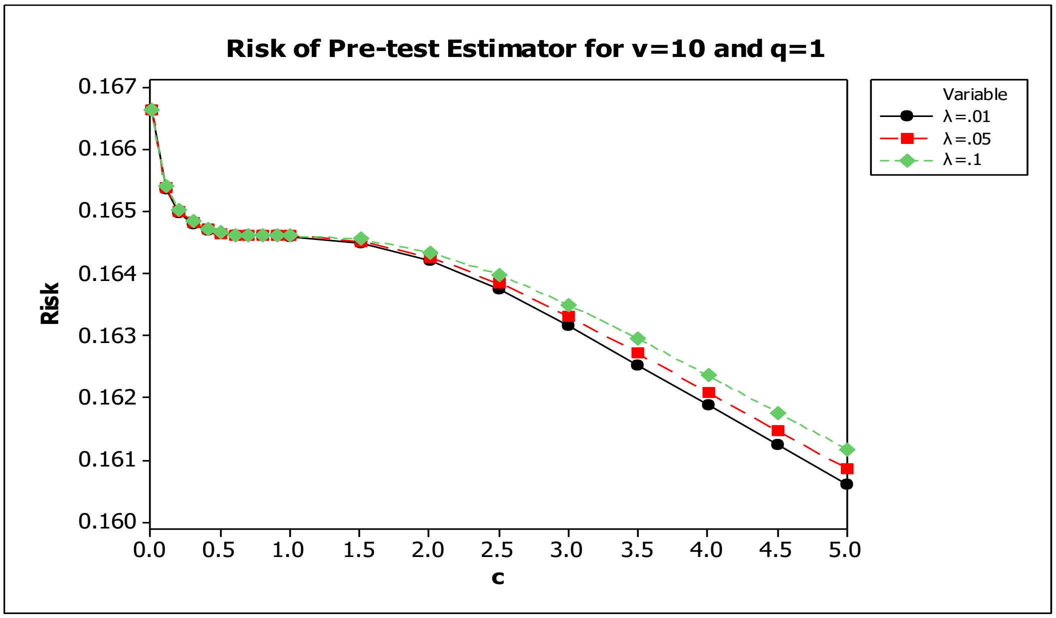


Table 2. Risk of Pre-test Estimator for selected values of $\sigma^2 = 1, v = 20, q, \lambda$ and c

	$v = 20$								
	c	0.01	0.05	0.1	0.25	0.5	1	5	10
$q = 1$	0.0	0.09091	0.09091	0.09091	0.09091	0.09091	0.09091	0.09091	0.09091
	0.1	0.09048	0.09049	0.09050	0.09052	0.09056	0.09063	0.09086	0.09090
	0.5	0.09023	0.09023	0.09024	0.09025	0.09027	0.09033	0.09072	0.09088
	1.0	0.09021	0.09021	0.09021	0.09021	0.09022	0.09026	0.09065	0.09086
	1.5	0.09019	0.09019	0.09020	0.09023	0.09027	0.09036	0.09083	0.09093
	2.0	0.09011	0.09012	0.09015	0.09022	0.09033	0.09053	0.09130	0.09116
	2.5	0.08997	0.09000	0.09004	0.09016	0.09036	0.09074	0.09209	0.09163
	3.0	0.08979	0.08984	0.08990	0.09007	0.09037	0.09094	0.09318	0.09242
	3.5	0.08958	0.08964	0.08972	0.08995	0.09034	0.09111	0.09455	0.09357
	4.0	0.08937	0.08944	0.08953	0.08980	0.09028	0.09124	0.09616	0.09514
	4.5	0.08915	0.08923	0.08933	0.08964	0.09018	0.09133	0.09795	0.09715
	5.0	0.08894	0.08903	0.08913	0.08947	0.09007	0.09138	0.09987	0.09960
$q = 2$	0.0	0.09091	0.09091	0.09091	0.09091	0.09091	0.09091	0.09091	0.09091
	0.1	0.09027	0.09029	0.09030	0.09033	0.09039	0.09049	0.09083	0.09090
	0.5	0.08925	0.08926	0.08927	0.08930	0.08936	0.08949	0.09040	0.09081
	1.0	0.08915	0.08915	0.08915	0.08916	0.08918	0.08926	0.09016	0.09073
	1.5	0.08903	0.08905	0.08907	0.08913	0.08924	0.08947	0.09085	0.09113
	2.0	0.08865	0.08869	0.08874	0.08890	0.08917	0.08971	0.09259	0.09251
	2.5	0.08807	0.08813	0.08821	0.08846	0.08890	0.08981	0.09521	0.09523
	3.0	0.08741	0.08750	0.08760	0.08792	0.08849	0.08973	0.09842	0.09949
	3.5	0.08677	0.08686	0.08697	0.08734	0.08800	0.08951	0.10190	0.10525
	4.0	0.08618	0.08627	0.08639	0.08677	0.08749	0.08919	0.10541	0.11234
	4.5	0.08567	0.08576	0.08587	0.08625	0.08699	0.08881	0.10876	0.12049
	5.0	0.08524	0.08532	0.08543	0.08579	0.08653	0.08842	0.11182	0.12939
$q = 5$	0.0	0.09091	0.09091	0.09091	0.09091	0.09091	0.09091	0.09091	0.09091
	0.1	0.09056	0.09056	0.09057	0.09059	0.09062	0.09068	0.09087	0.09090
	0.5	0.08676	0.08678	0.08681	0.08689	0.08704	0.08734	0.08940	0.09052
	1.0	0.08604	0.08604	0.08604	0.08606	0.08610	0.08623	0.08822	0.08999
	1.5	0.08523	0.08526	0.08532	0.08547	0.08575	0.08637	0.09120	0.09329
	2.0	0.08311	0.08319	0.08329	0.08360	0.08417	0.08544	0.09665	0.10281
	2.5	0.08073	0.08082	0.08094	0.08132	0.08203	0.08370	0.10203	0.11730
	3.0	0.07872	0.07881	0.07892	0.07929	0.08002	0.08183	0.10608	0.13386
	3.5	0.07724	0.07731	0.07740	0.07773	0.07840	0.08018	0.10865	0.14999
	4.0	0.07620	0.07626	0.07633	0.07661	0.07720	0.07888	0.11003	0.16419
	4.5	0.07550	0.07554	0.07560	0.07583	0.07635	0.07790	0.11063	0.17589
	5.0	0.07503	0.07506	0.07511	0.07530	0.07576	0.07720	0.11075	0.18512

Figure 1(B) Risk of Pre-test Estimator for $\sigma^2 = 1, v = 10$ and $q = 1$

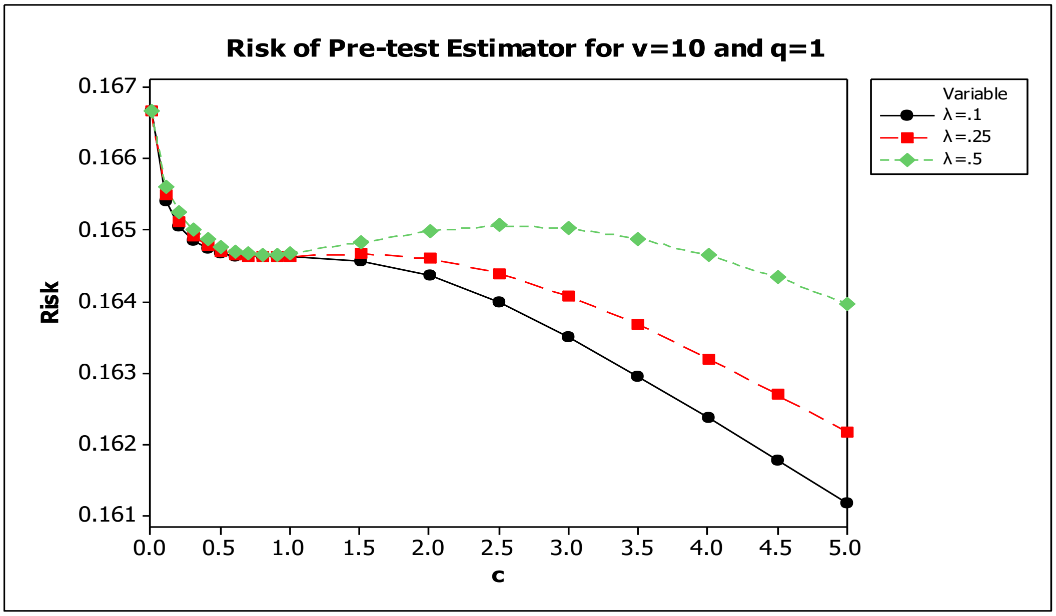


Figure 1(C) Risk of Pre-test Estimator for $\sigma^2 = 1, v = 10$ and $q = 1$

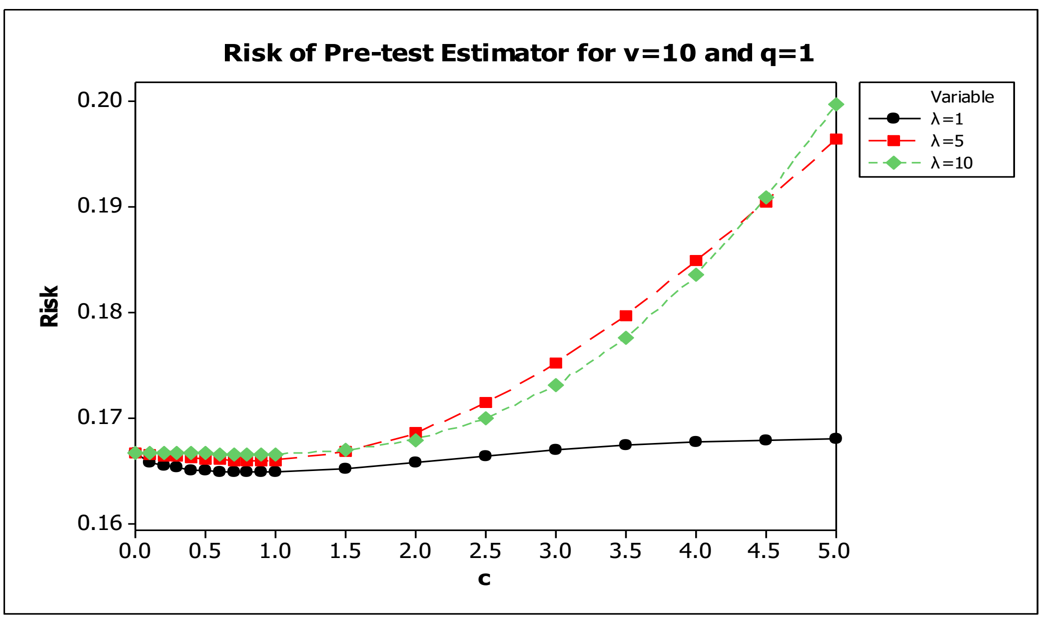


Table 3. Risk of Pre-test Estimator for selected values of $\sigma^2 = 1, v = 50, q, \lambda$ and c

$v = 50$									
	c	0.01	0.05	0.1	0.25	0.5	1	5	10
$q = 1$	0.0	0.03846	0.03846	0.03846	0.03846	0.03846	0.03846	0.03846	0.03846
	0.1	0.03838	0.03838	0.03838	0.03839	0.03839	0.03841	0.03845	0.03846
	0.5	0.03833	0.03833	0.03833	0.03833	0.03834	0.03835	0.03843	0.03846
	1.0	0.03832	0.03832	0.03832	0.03832	0.03833	0.03833	0.03841	0.03845
	1.5	0.03832	0.03832	0.03832	0.03833	0.03833	0.03835	0.03844	0.03846
	2.0	0.03831	0.03831	0.03831	0.03833	0.03835	0.03838	0.03851	0.03850
	2.5	0.03828	0.03829	0.03830	0.03832	0.03835	0.03842	0.03865	0.03857
	3.0	0.03825	0.03826	0.03827	0.03830	0.03836	0.03846	0.03884	0.03869
	3.5	0.03821	0.03822	0.03824	0.03828	0.03835	0.03850	0.03908	0.03887
	4.0	0.03817	0.03818	0.03820	0.03825	0.03834	0.03853	0.03938	0.03913
	4.5	0.03813	0.03814	0.03816	0.03822	0.03833	0.03855	0.03971	0.03946
	5.0	0.03809	0.03810	0.03812	0.03819	0.03831	0.03856	0.04007	0.03987
$q = 2$	0.0	0.03846	0.03846	0.03846	0.03846	0.03846	0.03846	0.03846	0.03846
	0.1	0.03834	0.03834	0.03835	0.03835	0.03836	0.03838	0.03845	0.03846
	0.5	0.03813	0.03813	0.03814	0.03814	0.03816	0.03818	0.03837	0.03844
	1.0	0.03810	0.03811	0.03811	0.03811	0.03811	0.03813	0.03831	0.03843
	1.5	0.03809	0.03809	0.03809	0.03810	0.03812	0.03816	0.03842	0.03848
	2.0	0.03802	0.03803	0.03803	0.03806	0.03812	0.03822	0.03872	0.03870
	2.5	0.03791	0.03792	0.03793	0.03798	0.03807	0.03825	0.03922	0.03915
	3.0	0.03777	0.03779	0.03781	0.03788	0.03799	0.03824	0.03985	0.03988
	3.5	0.03764	0.03766	0.03768	0.03776	0.03789	0.03820	0.04057	0.04093
	4.0	0.03752	0.03754	0.03757	0.03764	0.03779	0.03814	0.04131	0.04227
	4.5	0.03742	0.03744	0.03746	0.03754	0.03769	0.03806	0.04204	0.04387
	5.0	0.03734	0.03735	0.03737	0.03745	0.03759	0.03798	0.04271	0.04567
$q = 5$	0.0	0.03846	0.03846	0.03846	0.03846	0.03846	0.03846	0.03846	0.03846
	0.1	0.03840	0.03840	0.03840	0.03840	0.03841	0.03842	0.03845	0.03846
	0.5	0.03760	0.03760	0.03761	0.03763	0.03767	0.03774	0.03818	0.03840
	1.0	0.03739	0.03739	0.03739	0.03739	0.03740	0.03743	0.03788	0.03827
	1.5	0.03724	0.03725	0.03726	0.03730	0.03735	0.03748	0.03845	0.03881
	2.0	0.03678	0.03680	0.03682	0.03689	0.03702	0.03731	0.03971	0.04070
	2.5	0.03625	0.03627	0.03629	0.03638	0.03654	0.03693	0.04107	0.04394
	3.0	0.03581	0.03583	0.03586	0.03594	0.03609	0.03650	0.04214	0.04796
	3.5	0.03552	0.03553	0.03555	0.03562	0.03576	0.03614	0.04281	0.05207
	4.0	0.03533	0.03534	0.03536	0.03541	0.03553	0.03588	0.04315	0.05576
	4.5	0.03523	0.03523	0.03524	0.03528	0.03538	0.03570	0.04326	0.05879
	5.0	0.03516	0.03517	0.03518	0.03521	0.03529	0.03558	0.04325	0.06109

Figure 2(A) Risk of Pre-test Estimator for $\sigma^2 = 1$, $v = 50$ and $q = 1$

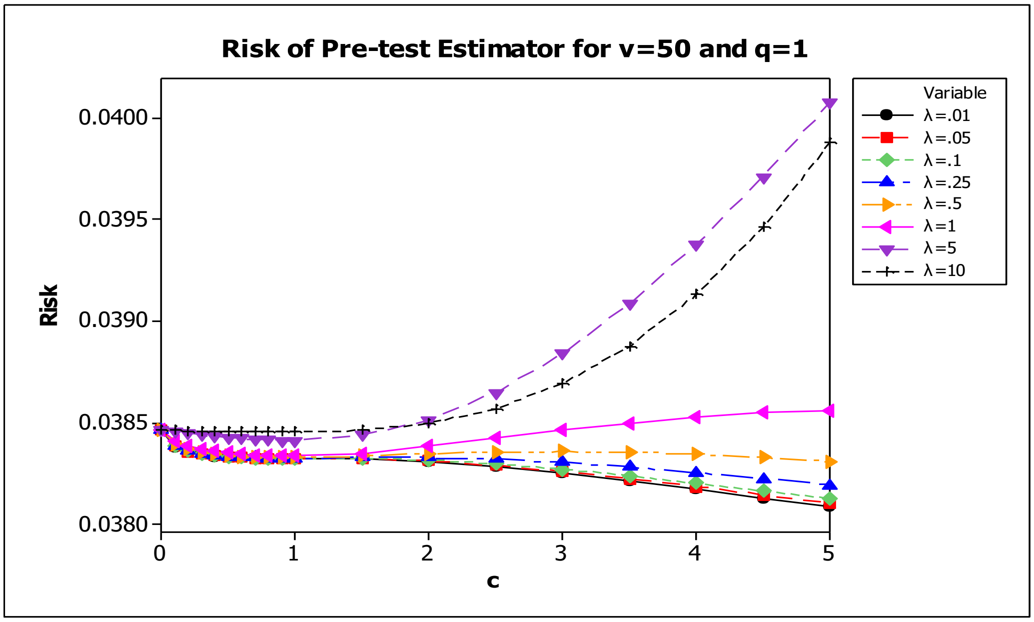


Figure 2(B) Risk of Pre-test Estimator for $\sigma^2 = 1, v = 50$ and $q = 1$

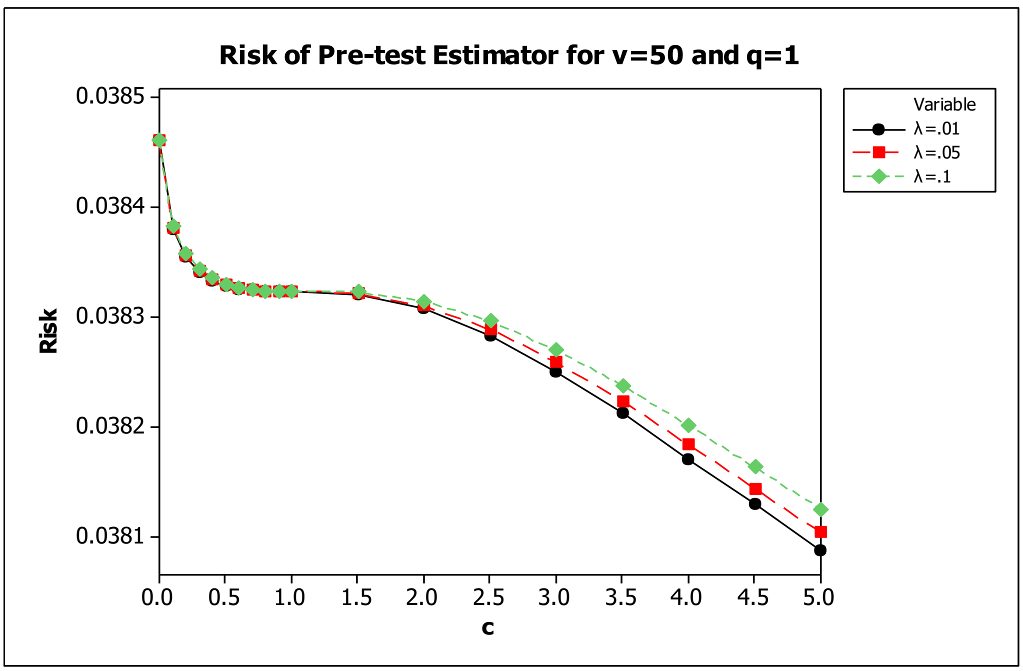
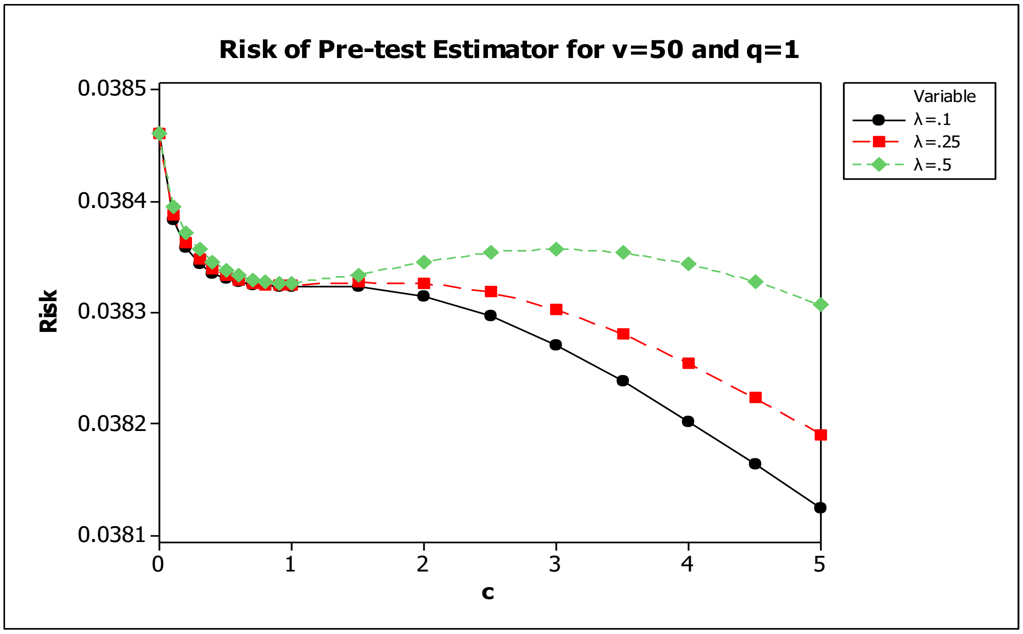


Figure 2(C) Risk of Pre-test Estimator for $\sigma^2 = 1, v = 10$ and $q = 1$



The effect of the increase in the degrees of freedom are presented in the Tables 2 and 3 and presented graphically for $v = 50$ in Figures 2 (A, B, C). The effect of degrees of freedom on behavior of Pre-test estimator for fixed value of non-centrality parameter λ and the critical value c, it can be seen from Tables 1 to 3, that when degrees of freedom increases the risk of Pre-test estimator decreases. The Tables 1 to 3 also provide the effect of number of restrictions on the risk performance of the Pre-test estimator of the disturbance variance. It is observed that when the number of restrictions on parameter increases the risk of the Pre-test decreases monotonically. Therefore, in the presence of prior information the Pre-test estimator gives better results. These results are also in agreement to what it is obtained using probability distribution.

Appendix

Proof of Theorem 1:

From the density function of the Pre-test estimator, all the moments of s_{PT}^2 can be obtained using

$$E[s_{PT}^2]^j = \int_0^\infty (s_{PT}^2)^j \, f(s_{PT}^2) ds_{PT}^2$$

Therefore, the j^{th} moment of the Pre-test estimator is obtained using density function (2.12) as

$$E[s_{PT}^2]^j = \sum_{i=0}^\infty w_i(\lambda) \left(\frac{m'}{2\,\sigma^2}\right)^{\frac{v+q}{2}+i} \frac{1}{\Gamma\left(\frac{v+q}{2}+i\right)} \cdot \mathrm{I}_{\frac{qc}{v+qc}}\left(\frac{q}{2}+i,\ \frac{v}{2}\right) \int_0^\infty \tau^{\frac{v+q}{2}+i+j-1}\, e^{-\left(\frac{m'}{2\,\sigma^2}\right)\tau} d\tau +$$
$$\sum_{i=0}^\infty w_i(\lambda) \frac{1}{\Gamma\left(\frac{q}{2}+i\right)\Gamma\left(\frac{v}{2}\right)} \left(\frac{m}{2\,\sigma^2}\right)^{\frac{v+q}{2}+i} \int_{\frac{qc}{v+qc}}^1 \frac{g^{\frac{q}{2}+i-1}}{(1-g)^{\frac{q}{2}+i+1}} \int_0^\infty \tau^{\frac{v+q}{2}+i+j-1}\, e^{-\left(\frac{m}{2\,\sigma^2(1-g)}\right)\tau} d\tau\, dg$$

On solving the integral parts

$$E[s_{PT}^2]^j = \sum_{i=0}^{\infty} w_i(\lambda) \left(\frac{2\,\sigma^2}{m'}\right)^j \frac{\Gamma\left(\frac{v+q}{2}+i+j\right)}{\Gamma\left(\frac{v+q}{2}+i\right)} . \mathrm{I}_{\frac{qc}{v+qc}}\left(\frac{q}{2}+i,\ \frac{v}{2}\right)$$

$$+ \sum_{i=0}^{\infty} w_i(\lambda) \frac{\Gamma\left(\frac{v+q}{2}+i+j\right)}{\Gamma\left(\frac{q}{2}+i\right)\Gamma\left(\frac{v}{2}\right)} \left(\frac{2\,\sigma^2}{m}\right)^j \int_{\frac{qc}{v+qc}}^{1} g^{\frac{q}{2}+i-1}\ (1-g)^{\frac{v}{2}+j-1} dg$$

the above equation may be written

$$E[s_{PT}^2]^j = \sum_{i=0}^{\infty} w_i(\lambda) \left(\frac{2\,\sigma^2}{m'}\right)^j \frac{\Gamma\left(\frac{v+q}{2}+i+j\right)}{\Gamma\left(\frac{v+q}{2}+i\right)} . \mathrm{I}_{\frac{qc}{v+qc}}\left(\frac{q}{2}+i,\ \frac{v}{2}\right) +$$

$$\sum_{i=0}^{\infty} w_i(\lambda) \frac{\Gamma\left(\frac{v+q}{2}+i+j\right)}{\Gamma\left(\frac{q}{2}+i\right)\Gamma\left(\frac{v}{2}\right)} \left(\frac{2\,\sigma^2}{m}\right)^j \left[\int_0^1 g^{\frac{q}{2}+i-1}\ (1-g)^{\frac{v}{2}+j-1} dg - \int_0^{\frac{qc}{v+qc}} g^{\frac{q}{2}+i-1}\ (1-g)^{\frac{v}{2}+j-1} dg\right]$$

Now using the beta function and incomplete beta function in above equation, obtain the j^{th} moment of the Pre-test estimator as given in Theorem 1.

Proof of the Theorem 2:

The bias and the risk of the Pre-test estimator under quadratic error loss function is given by

$$B[s_{PT}^2\ ] = E[s_{PT}^2] - \sigma^2 \tag{A.1}$$

$$R[s_{PT}^2\ ] = \ E[s_{PT}^2]^2 - 2\sigma^2 E[s_{PT}^2] + \sigma^4 \tag{A.2}$$

In order to find the bias and the risk of the Pre-test estimator let us substitute $j = 1$ and $j = 2$ in (3.1) as

$$E[s_{PT}^2\ ] = \ \left(\frac{2\,\sigma^2}{m'}\right) \sum_{i=0}^{\infty} w_i(\lambda)\ \frac{\Gamma\left(\frac{v+q}{2}+i+1\right)}{\Gamma\left(\frac{v+q}{2}+i\right)} . \mathrm{I}_{\frac{qc}{v+qc}}\left(\frac{q}{2}+i,\ \frac{v}{2}\right)$$

$$+ \left(\frac{2\,\sigma^2}{m'}\right) \sum_{i=0}^{\infty} w_i(\lambda) \frac{\Gamma\left(\frac{v}{2}+1\right)}{\Gamma\left(\frac{v}{2}\right)} \ \left[1 - \mathrm{I}_{\frac{qc}{v+qc}}\left(\frac{q}{2}+i,\ \frac{v}{2}+1\right)\right] \tag{A.3}$$

$$E[s_{PT}^2\ ]^2 = \ \left(\frac{2\,\sigma^2}{m'}\right)^2 \sum_{i=0}^{\infty} w_i(\lambda)\ \frac{\Gamma\left(\frac{v+q}{2}+i+2\right)}{\Gamma\left(\frac{v+q}{2}+i\right)} . \mathrm{I}_{\frac{qc}{v+qc}}\left(\frac{q}{2}+i,\ \frac{v}{2}\right)$$

$$+ \left(\frac{2\,\sigma^2}{m'}\right)^2 \sum_{i=0}^{\infty} w_i(\lambda) \frac{\Gamma\left(\frac{v}{2}+2\right)}{\Gamma\left(\frac{v}{2}\right)} \ \left[1 - \mathrm{I}_{\frac{qc}{v+qc}}\left(\frac{q}{2}+i,\ \frac{v}{2}+2\right)\right] \tag{A.4}$$

Therefore, by substituting the value of (A.3) in (A.1) obtain the bias of the Pre-test estimator and by substituting the values of (A.3) and (A.4) in (A.2) the risk of the Pre-test estimator under quadratic error loss be obtained, as given in Theorem 2.

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